Modal logic axioms valid in quotient spaces of finite CW-complexes and other families of topological spaces

Maria Nogin
Bing Xu

California State University, Fresno

AMS/MAA Joint Mathematics Meetings
Atlanta, GA
January 5, 2017
• Definitions
  ▷ Topology
    ★ Particular point topological space
    ★ Excluded point topological space
Outline

• Definitions
  ▶ Topology
    ★ Particular point topological space
    ★ Excluded point topological space
    ★ Quotient space of a CW-complex
Outline

- Definitions
  - Topology
    - Particular point topological space
    - Excluded point topological space
    - Quotient space of a CW-complex
  - Logic
    - Modal logic language $\mathcal{L}\Box$
    - Interpretation of $\mathcal{L}\Box$ in topological spaces
• Definitions
  ▶ Topology
    ★ Particular point topological space
    ★ Excluded point topological space
    ★ Quotient space of a CW-complex
  ▶ Logic
    ★ Modal logic language $\mathcal{L}_\Box$
    ★ Interpretation of $\mathcal{L}_\Box$ in topological spaces
    ★ Some axioms: $K$, $T$, $4$, $M$, $G$
• Definitions
  ▶ Topology
    ★ Particular point topological space
    ★ Excluded point topological space
    ★ Quotient space of a CW-complex
  ▶ Logic
    ★ Modal logic language $\mathcal{L}_{\Box}$
    ★ Interpretation of $\mathcal{L}_{\Box}$ in topological spaces
    ★ Some axioms: $K, T, 4, M, G$

• Validity of axioms $M$ and $G$ in
• Definitions
  ▶ Topology
    ★ Particular point topological space
    ★ Excluded point topological space
    ★ Quotient space of a CW-complex
  ▶ Logic
    ★ Modal logic language $\mathcal{L}_\Box$
    ★ Interpretation of $\mathcal{L}_\Box$ in topological spaces
    ★ Some axioms: $K$, $T$, $4$, $M$, $G$

• Validity of axioms $M$ and $G$ in
  ▶ particular point topological spaces
  ▶ excluded point topological spaces
  ▶ quotient spaces of finite CW-complexes
Definition 1
Let $X$ be any nonempty set and $p \in X$. The collection

$$T_p = \{S \subseteq X \mid p \in S \text{ or } S = \emptyset\}$$

of subsets of $X$ is called the \textit{particular point topology} on $X$. 
Definition 1
Let $X$ be any nonempty set and $p \in X$. The collection

$$T_p = \{ S \subseteq X \mid p \in S \text{ or } S = \emptyset \}$$

of subsets of $X$ is called the particular point topology on $X$.

Definition 2
Let $X$ be any nonempty set and $e \in X$. The collection

$$T_e = \{ S \subseteq X \mid e \notin S \text{ or } S = X \}$$

of subsets of $X$ is called the excluded point topology on $X$. 
Quotient space of a CW-complex

**Definition 3**

Let $X$ be a CW-complex. Its quotient space $Q(X)$ is a topological space whose points are in one-to-one correspondence with cells of $X$, and a subset of $Q(X)$ is open if and only if the union of the corresponding cells is open in $X$. 

**Example:** The quotient space of the standard CW-complex of $\mathbb{R}P^1$ is the Sierpinski space.

**Definition 4**

If $X$ is a CW-complex, its cell $c$ is called a top cell if it is not in the boundary of any other cell. A single point in $Q(X)$ is open if and only if it corresponds to a top cell.
**Definition 3**

Let $X$ be a CW-complex. Its quotient space $Q(X)$ is a topological space whose points are in one-to-one correspondence with cells of $X$, and a subset of $Q(X)$ is open if and only if the union of the corresponding cells is open in $X$.

Example: The quotient space of the standard CW-complex of $\mathbb{R}P_1$ is the Sierpinski space.
Definition 3
Let $X$ be a CW-complex. Its quotient space $Q(X)$ is a topological space whose points are in one-to-one correspondence with cells of $X$, and a subset of $Q(X)$ is open if and only if the union of the corresponding cells is open in $X$.

Example: The quotient space of the standard CW-complex of $\mathbb{R}P_1$ is the Sierpinski space.

Definition 4
If $X$ is a CW-complex, its cell $c$ is called a top cell if it is not in the boundary of any other cell.
Definition 3

Let $X$ be a CW-complex. Its quotient space $Q(X)$ is a topological space whose points are in one-to-one correspondence with cells of $X$, and a subset of $Q(X)$ is open if and only if the union of the corresponding cells is open in $X$.

Example: The quotient space of the standard CW-complex of $\mathbb{R}P_1$ is the Sierpinski space.

Definition 4

If $X$ is a CW-complex, its cell $c$ is called a top cell if it is not in the boundary of any other cell.

A single point in $Q(X)$ is open if and only if it corresponds to a top cell.
Definition 5

- $\mathcal{L} \Box$ is the modal logic language consisting of propositional variables, $\land$, $\lor$, $\neg$, and $\Box$. Then, $\Diamond$ is defined by $\Diamond P = \neg \Box \neg P$. 

Interpretation of $\mathcal{L} \Box$ in topological spaces
Interpretation of $\mathcal{L}^{\Box}$ in topological spaces

**Definition 5**

- $\mathcal{L}^{\Box}$ is the modal logic language consisting of propositional variables, $\land$, $\lor$, $\neg$, and $\Box$. Then, $\lozenge$ is defined by $\lozenge P = \neg \Box \neg P$.

- A topological model of $\mathcal{L}^{\Box}$ is a pair $\langle X, \| \| \rangle$, where
  1. $X$ is a topological space, and
  2. $\| \cdot \|$ is a function mapping formulas in $\mathcal{L}^{\Box}$ to subsets of $X$ satisfying
     \[
     \| F \land G \| = \| F \| \cap \| G \|
     \]
     \[
     \| F \lor G \| = \| F \| \cup \| G \|
     \]
     \[
     \| \neg F \| = \| F \|^{C} = X \setminus \| F \|
     \]
     \[
     \| \Box F \| = \text{int}(\| F \|)
     \]
Interpretation of $\mathcal{L}_\square$ in topological spaces

**Definition 5**

- $\mathcal{L}_\square$ is the modal logic language consisting of propositional variables, $\land$, $\lor$, $\neg$, and $\square$. Then, $\lozenge$ is defined by $\lozenge P = \neg \square \neg P$.

- A **topological model** of $\mathcal{L}_\square$ is a pair $\langle X, \| \| \rangle$, where
  1. $X$ is a topological space, and
  2. $\| \cdot \|$ is a function mapping formulas in $\mathcal{L}_\square$ to subsets of $X$ satisfying
     
     \[
     \| F \land G \| = \| F \| \cap \| G \|
     \]
     
     \[
     \| F \lor G \| = \| F \| \cup \| G \|
     \]
     
     \[
     \| \neg F \| = \| F \|^C = X \setminus \| F \|
     \]
     
     \[
     \| \square F \| = \text{int}(\| F \|)
     \]

- A formula $F$ is called *valid* in a topological space $X$ if for any topological model $\langle X, \| \| \rangle$, we have $\| F \| = X$. 
Axioms

S4:

- All axioms of the classical propositional logic,
- Axiom K: $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$,
- Axiom T: $\Box A \rightarrow A$,
- Axiom 4: $\Box A \rightarrow \Box \Box A$, 
Axioms

S4:

- All axioms of the classical propositional logic,
- Axiom K: $\Box(A \to B) \to (\Box A \to \Box B)$,
- Axiom T: $\Box A \to A$,
- Axiom 4: $\Box A \to \Box \Box A$,

Inference rules

- Modus ponens: $A, A \to B \quad \frac{}{B}$,
- Necessitation: $A \quad \frac{}{\Box A}$.
Axioms

**S4:**

- All axioms of the classical propositional logic,
- Axiom K: $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$,
- Axiom T: $\Box A \rightarrow A$,
- Axiom 4: $\Box A \rightarrow \Box \Box A$,

**Inference rules**

- Modus ponens: \[
\frac{A, A \rightarrow B}{B}
\]

- Necessitation: \[
\frac{A}{\Box A}
\]

**Axiom M:** \[
(\Box \Diamond P) \rightarrow (\Diamond \Box P) \equiv (\Diamond \Box P) \lor (\Diamond \Box \neg P)
\]
Axioms

S4:

• All axioms of the classical propositional logic,
• Axiom K: □(A → B) → (□A → □B),
• Axiom T: □A → A,
• Axiom 4: □A → □□A,

Inference rules

• Modus ponens: \[ \frac{A, A \to B}{B} \],
• Necessitation: \[ \frac{A}{\Box A} \].

Axiom M: \( (\Box \Diamond P) \to (\Diamond \Box P) \equiv (\Diamond \Box P) \lor (\Diamond \Box \neg P) \)

Axiom G: \( (\Diamond \Box P) \to (\Box \Diamond P) \equiv (\Box \Diamond P) \lor (\Box \Diamond \neg P) \)
$M$ and $G$ in particular point and excluded point topological spaces

**Theorem 6**

*Both axioms $M$ and $G$ are valid in any particular point topological space.*
$M$ and $G$ in particular point and excluded point topological spaces

**Theorem 6**

Both axioms $M$ and $G$ are valid in any particular point topological space.

**Theorem 7**

1. Axiom $M$ is valid in any excluded point topological space.
2. Axiom $G$ is valid in an excluded point topological space if and only if the space has only 1 or 2 points.
Theorem 8

(1) Axiom $M$ is valid in the quotient space of any finite CW-complex.

(2) Axiom $G$ is valid in the quotient space of a finite CW-complex iff each connected component of the CW-complex has a unique top cell.
Theorem 8

(1) Axiom \( M \) is valid in the quotient space of any finite CW-complex.

(2) Axiom \( G \) is valid in the quotient space of a finite CW-complex iff each connected component of the CW-complex has a unique top cell.

Idea of proof:

(1) For any \( c \in Q(X) \), \( \exists t \in Q(X) \) corresponding to a top cell s.t. \( c \in \text{cl}(\{t\}) = \text{cl}(\text{int}(\{t\})) \).

(2) \( \Leftarrow \) If \( t \) corresponds to a top cell, then its entire connected component is in \( \text{int}(\text{cl}(\{t\})) \).

(2) \( \Rightarrow \) If a connected component contains more than one top cell, there are two top cells whose closures have non-empty intersection. Let \( t_1, t_2 \in Q(X) \) correspond to such two top cells and let \( c \) correspond to a cell in the intersection of their closures. Consider a validation mapping such that \( t_1 \in ||P|| \) and \( t_2 \not\in ||P|| \). Then \( c \not\in ||\square\diamond P|| \) and \( c \not\in ||\square\diamond \neg P|| \).
Theorem 8

(1) Axiom $M$ is valid in the quotient space of any finite CW-complex.

(2) Axiom $G$ is valid in the quotient space of a finite CW-complex iff each connected component of the CW-complex has a unique top cell.

Idea of proof:

(1) For any $c \in Q(X)$, $\exists t \in Q(X)$ corresponding to a top cell s.t. $c \in \text{cl}(\{t\}) = \text{cl}(\text{int}(\{t\}))$.

(2) ($\Leftarrow$) If $t$ corresponds to a top cell, then its entire connected component is in $\text{int}(\text{cl}(\{t\}))$. 
Theorem 8

(1) Axiom $M$ is valid in the quotient space of any finite CW-complex.

(2) Axiom $G$ is valid in the quotient space of a finite CW-complex iff each connected component of the CW-complex has a unique top cell.

Idea of proof:

(1) For any $c \in Q(X)$, $\exists t \in Q(X)$ corresponding to a top cell s.t. $c \in \text{cl}({t}) = \text{cl}(\text{int}({t}))$.

(2) ($\Leftarrow$) If $t$ corresponds to a top cell, then its entire connected component is in $\text{int}(\text{cl}({t}))$.

($\Rightarrow$) If a connected component contains more than one top cell, there are two top cells whose closures have non-empty intersection.
Theorem 8

(1) Axiom $M$ is valid in the quotient space of any finite CW-complex.

(2) Axiom $G$ is valid in the quotient space of a finite CW-complex iff each connected component of the CW-complex has a unique top cell.

Idea of proof:

(1) For any $c \in Q(X)$, $\exists t \in Q(X)$ corresponding to a top cell s.t. $c \in \text{cl}({t}) = \text{cl}(	ext{int}({t}))$.

(2) $(\Leftarrow)$ If $t$ corresponds to a top cell, then its entire connected component is in $\text{int}(\text{cl}({t}))$.

$(\Rightarrow)$ If a connected component contains more than one top cell, there are two top cells whose closures have non-empty intersection. Let $t_1, t_2 \in Q(X)$ correspond to such two top cells and let $c$ correspond to a cell in the intersection of their closures.
**Theorem 8**

1. Axiom $M$ is valid in the quotient space of any finite CW-complex.
2. Axiom $G$ is valid in the quotient space of a finite CW-complex iff each connected component of the CW-complex has a unique top cell.

Idea of proof:

1. For any $c \in Q(X)$, $\exists t \in Q(X)$ corresponding to a top cell s.t. $c \in \text{cl}\{\{t\}\} = \text{cl}(\text{int}(\{t\}))$.

2. $(\Leftarrow)$ If $t$ corresponds to a top cell, then its entire connected component is in $\text{int}(\text{cl}(\{t\}))$.

$\Rightarrow$ If a connected component contains more than one top cell, there are two top cells whose closures have non-empty intersection. Let $t_1, t_2 \in Q(X)$ correspond to such two top cells and let $c$ correspond to a cell in the intersection of their closures. Consider a validation mapping such that $t_1 \in \|P\|$ and $t_2 \notin \|P\|$.
Theorem 8

(1) Axiom M is valid in the quotient space of any finite CW-complex.

(2) Axiom G is valid in the quotient space of a finite CW-complex iff each connected component of the CW-complex has a unique top cell.

Idea of proof:
(1) For any \( c \in Q(X) \), \( \exists t \in Q(X) \) corresponding to a top cell s.t. \( c \in \text{cl}(\{t\}) = \text{cl}(\text{int}(\{t\})) \).

(2) (\( \Leftarrow \)) If \( t \) corresponds to a top cell, then its entire connected component is in \( \text{int}(\text{cl}(\{t\})) \).

(\( \Rightarrow \)) If a connected component contains more than one top cell, there are two top cells whose closures have non-empty intersection. Let \( t_1, t_2 \in Q(X) \) correspond to such two top cells and let \( c \) correspond to a cell in the intersection of their closures.

Consider a validation mapping such that \( t_1 \in \|P\| \) and \( t_2 \notin \|P\| \). Then \( c \notin \|\Box\Diamond P\| \) and \( c \notin \|\Box\Diamond \neg P\| \).


Thank you!