# Chapter 2
# Fractions - Representation and Operations

## 2.1 TEACHING AND LEARNING FRACTIONS

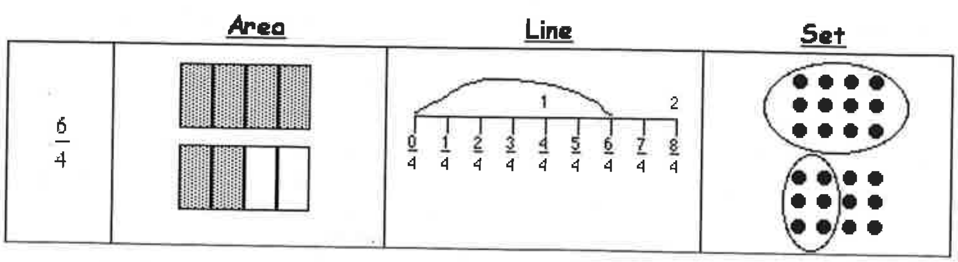
In the previous chapter, we saw how the same problem may be solved by using multiple approaches. Similarly, using multiple representations for fractions can provide valuable ways to better understand and convey concepts such as: common denominator, addition/subtraction and multiplication/division of fractions. This chapter investigates models for representations and operations with fractions so that as teachers, you will be able to understand the arithmetic of fractions at a deeper level yourselves. Hence, you will be able to instill good learning habits of mathematics into the practices of your students. In this chapter we will focus on two Standards for Mathematical Practice, namely Modeling with mathematics (Standard 4) and Attending to precision (Standard 6), in the context of fractions. Remember, if you can understand a subject, you will naturally find the means to teach it - understanding is infectious and can produce knowledge, confidence and enthusiasm greatly contributing to student engagement.

## 2.2 VISUAL MODELS FOR FRACTION REPRESENTATION

The three most often used visual models to represent fractions are:

A. Area Diagrams
B. Number Line Diagrams
C. Set Diagrams

These diagrams help students to visualize fractions to better help them understand models for operations conceptually, instead of just formulas, like $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, to be memorized. As the Common Core Standards emphasize fraction concepts in grades 3–5, with the assumption of background knowledge of 'whole number' concepts developed in grades 1 & 2, number line and area diagrams are essential to convey fraction arithmetic based on their connection to the whole number concept.




### Area Diagrams

As seen in the figure above, one way to employ area diagrams for visual representation of fractional quantities involves creating rectangles by dividing the rectangle into the same number of parts as the denominator, while shading a numerator number of those subdivisions. For example, we see to represent $\frac{6}{4}$, two rectangles are subdivided equally into four pieces each, making $\frac{1}{4}$'s, and then 6 of those pieces are shaded.

It is important to realize that area diagrams for visual fraction representation does not insist upon only using rectangles. Any shapes can be used, as long as they are subdivided into equal-area parts. For example, you could use the following manipulatives to represent fractions using shapes other than rectangles.

- Pattern Blocks
- Pie Charts/Spinners
- Geoboards

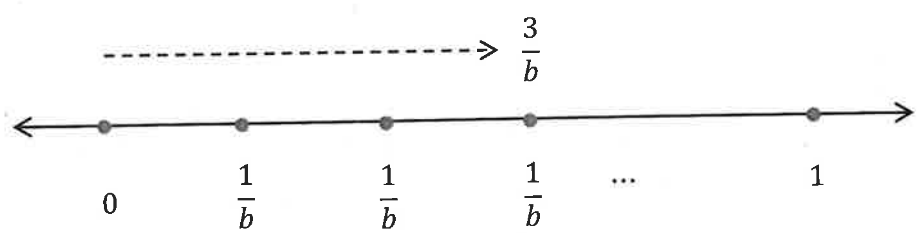
It will be important to distinguish between area diagrams for visual representation of fractions and the area model for fraction operations, as in section 2.4, the area model for multiplication and division of fractions will be introduced, which *specifically* requires the use of rectangles.

**Number Line Diagrams**

In our view, a central reason the Standards for Mathematical Practice emphasize the use of number line diagrams, is that it can apply to a wide variety of common core topics, such as:

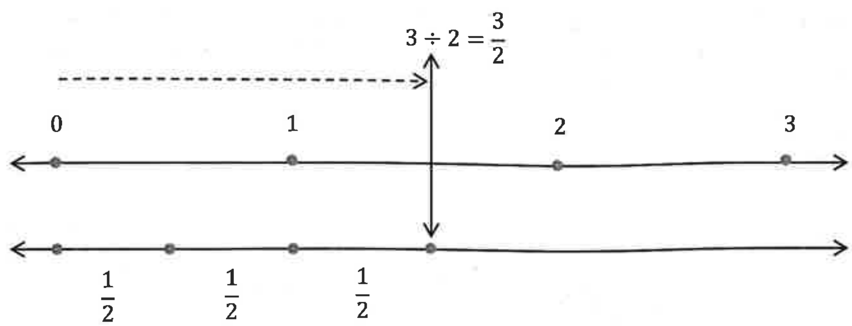
- Number ordering (greater than, less than, =, etc.)
- Rational numbers
- Decimals
- Approximate positions of irrational numbers ($\pi$, $\sqrt{2}$, etc.)

**Method 1: Subdivided Unit Interval**
Using the number line model to represent $\frac{3}{b}$ in a unit interval:
On the number line, the whole is the unit interval, which is the interval from 0 to 1, measured by length. To represent fractions such as $\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{b}$, the whole is divided into $b$ equal parts, so that each part has length $\frac{1}{b}$. We describe $\frac{3}{b}$ as represented in the number line as *three* of the $\frac{1}{b}$ lengths, as seen in the figure below

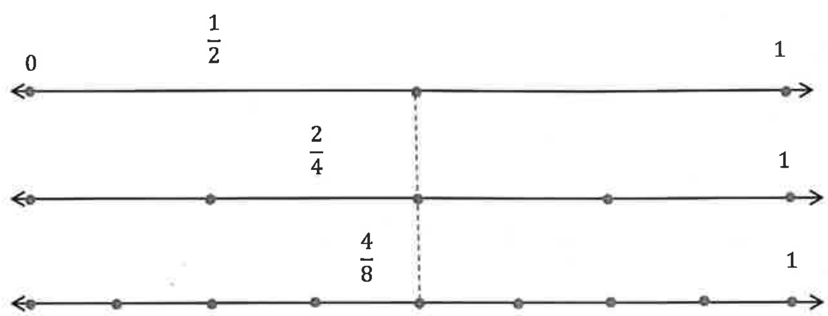



**Method 2: Any Subdivided Interval, or 'Quotient' Method**
The key idea for this variation of the number line model is to view $\frac{a}{b}$ not as the distance $\frac{1}{b}$ added up $a$-times; but, it interprets the representation $\frac{a}{b}$ as $a \div b$, which consists of the distance $a$ on the number line, subdivided into $b$ equal parts.

This might seem an insignificant difference, but it represents a different interpretation of the fraction since the previous method dealt with unit interval subdivisions; whereas, this method deals with the subdivisions over the entire numerator-length interval. In other words, to use the example of $\frac{3}{2}$; it is actually quite remarkable and significant that three $\frac{1}{2}$ subdivisions of the unit interval, is the same length as taking the interval of length 3, and dividing it into two pieces (see figure below). Why this works out to be equivalent, is the purpose of the models for fraction operations, which are closely connected to the models for fraction representations. Mathematical understanding is often the cumulative effect of many previous connected understandings, which can eventually contribute to an overall understanding; hence, it is recommended to use as many of the models and variants, such as methods 1 and 2, as possible in order to produce behaviors in students contributing to being flexible problem solvers.




Another important use of number line diagrams is to allow students to easily visualize the *equivalence* of reduced fractions and their non-reduced counterparts, as seen in the following diagram depicting the equivalence of $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$.




**Set Diagrams**

Set diagrams rely upon the use of individual objects which function as elements of a set making a 'whole', or 1. Since the entire collection of elements is a 'whole', then the fraction is the number of distinguished objects (numerator) 'of the whole' (denominator). In the above figure, as the circles are arranged into two groups of 12 pieces, then by segmenting the groups with ovals, the fraction $1\ \frac{6}{12}$ is represented, and can be concretely seen by the student that $1\ \frac{6}{12}$ must be equivalent to $1\ \frac{1}{2}$. Set diagrams can be used in any variety of ways using objects such as

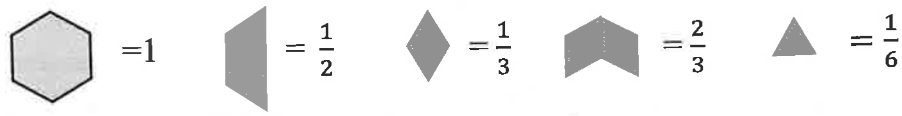
- Colored Chips
- Toys
- Buttons
- Coins
- Integer Chips

Understanding fractions and their operations has typically shown to present challenging tasks for students. Mastering the operations without conceptual understanding can be one of the main reasons for these difficulties.

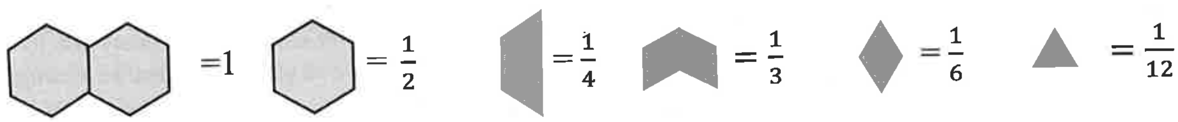
## 2.3 ADDITION AND SUBTRACTION OF FRACTIONS

In Common Core Standards-based instruction, students in grades 1 and 2 use the number line representation in order to make sense of addition and subtraction of whole numbers. The same representation, along with other visual representations, can be used to help learners construct viable arguments about the ways procedures should be generalized or modified for operations of fractions. For example, the need for a common denominator in adding or subtracting fractions arises in a natural way when number line, or pattern block, diagrams are used.

Based upon a variation of area diagrams for fraction representations, pattern blocks are geometric shapes which are subdivisions of the regular hexagon (a 6-sided polygon). For example, if we consider one hexagon to represent a whole (=1), then the values of pieces look like:




But, if we now consider two hexagons to represent a whole (=1), then the pieces look like:




In investigation 6 you will use number line diagrams, and in investigation 8 you will use pattern block diagrams for fraction addition and subtraction.

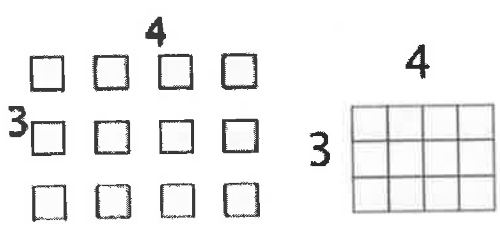
## 2.4 MULTIPLICATION OF FRACTIONS

### Fraction Multiplication Using Rectangles

The concept of multiplication of fractions is directly connected to the multiplication of whole numbers. For example, just using whole numbers we know that $3 \times 4$ means 3 groups of 4 or 4 groups of 3.

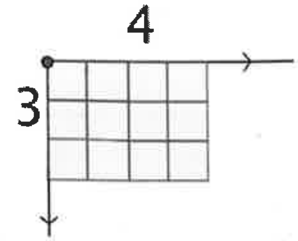
$$3 \times 4 = 4 + 4 + 4 = 3 + 3 + 3 + 3$$

We can easily represent this idea using visual models:




3 rows of 4 or as 4 columns of 3

This representation can be seen as a rectangle with side lengths of 3 and 4 (view this as a two number lines with lengths 3 and 4 as shown in the figure below). The answer for the multiplication is found as the area of the rectangle (number of units squares that cover the rectangle). In this case, the length of the rectangle is 3 units, the width is 4 units, and the area of the rectangle is 12 square units.




This idea can be used for any multiplication (whole numbers, polynomials, decimals or fractions), since multiplication is defined as repeated addition; therefore, when we can represent multiplication with a visual model using three simple rules (often called area model, box model, or array model).

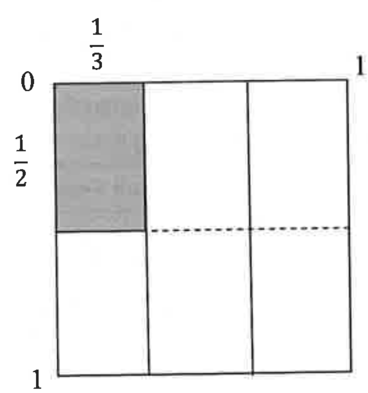
**Three rules for the visual area model for multiplication**

1. **Any multiplication can be represented as a rectangle**
2. **The two numbers/quantities that multiply represent the length and width of the rectangle.**
3. **The area of the rectangle represents the answer to the multiplication.**

We can see in the following example that the same concepts that applied to whole number multiplication, can also apply to fraction multiplication. To illustrate the multiplication of any two fractions, use the above three rules to find the area of a rectangle, made of unit squares, so that fractional side lengths can be created.

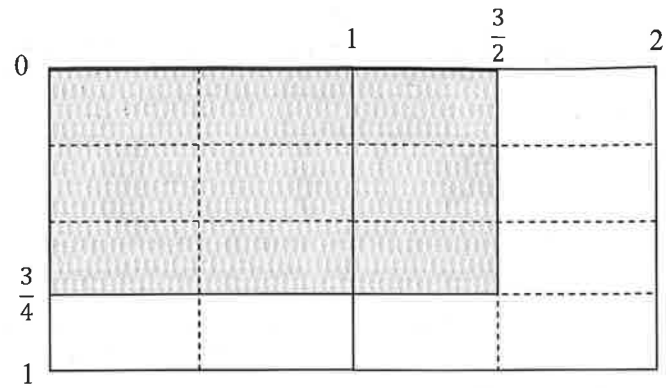
Keep in mind that the area model for fraction operations is a little different than the area model for fraction representation, in that the model for representations is concerned solely with shading in a subdivided whole with correct fractional amounts; while, the area model for fraction operations is done by finding the area of a rectangle, given two fractions represented on number lines.

*Example 2.1.* $\frac{1}{2} \times \frac{1}{3}$




Notice here that the edges are the fraction factors that are multiplying and that the answer is the shaded *area*, which must be interpreted in terms of how many parts of 'the whole', meaning the $1 \times 1$ square. In this case, then, the answer is $\frac{1}{6}$ because we see that the whole has been divided into six equal parts and the rectangle of dimensions $\frac{1}{2} \times \frac{1}{3}$ has an area of $\frac{1}{6}$.

*Example 2.2.* $\frac{3}{4} \times \frac{3}{2}$



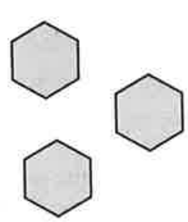
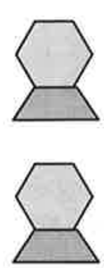
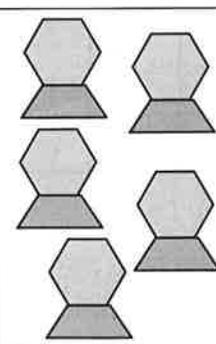
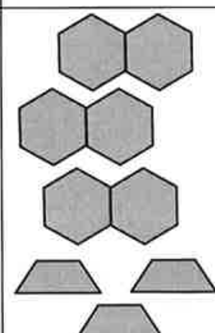

After finding the shaded rectangular region made by the $\frac{3}{4} \times \frac{3}{2}$ rectangle, we see that the area of the shaded region can be manipulated to fill in the two remaining squares on the bottom of the first $1 \times 1$ square and see that we have one small rectangle remaining, giving us the answer: $1\frac{1}{8}$ of a $1 \times 1$ square whole.

Fraction Multiplication Using Pattern Blocks

Using pattern blocks comprises another way to model fraction multiplication and is particularly useful for developing conceptual understanding for why the formula $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ works. The key to the pattern block approach for a problem such as $\frac{1}{2} \times \frac{1}{3}$ is to view $\frac{1}{2}$ groups of $\frac{1}{3}$ as *one of the two groups* of $\frac{1}{3}$. To do this with pattern blocks, though, it is necessary to first make *one of the two groups* of $\frac{1}{3}$ by dividing the $\frac{1}{3}$ pattern block representation into two equivalent groups of blocks. Finally, *one* of these two groups is taken which, in this case, ends up being $\frac{1}{6}$ relative to the given whole. This process is illustrated visually in the next example, and also in Investigation 9.

*Example 2.3.* Given that two hexagons is the whole ($= 1$), we want to use pattern blocks to diagram the multiplication

$$\frac{5}{2} \times \frac{3}{2}$$

| Initial Drawing | Denominator Drawing | Numerator Drawing | Reduced Drawing | Answer |
|---|---|---|---|---|
|  |  |  |  | $3\frac{3}{4}$ |

**(a)** The initial drawing shows $\frac{3}{2}$.
**(b)** The denominator drawing breaks $\frac{3}{2}$ in two parts due to the denominator of $\frac{5}{2}$ consisting of a two.
**(c)** Next, the numerator drawing depicts taking five (5) of those individual two pieces.
**(d)** For the final reduced drawing, one gathers up all of the 'wholes' and makes the remaining pieces all of one 'largest' color (the reduction).

## 2.5 DIVISION OF FRACTIONS

In the book, *Knowing and Teaching Elementary Mathematics* (1990), the mathematics education researcher, Liping Ma, shows how Chinese elementary school teachers develop their highly successful conceptual approaches to teaching. For the teaching of the topic of division with fractions, three basic models are employed:

1. measurement model
2. partitive model
3. factors and product (area) model

**Measurement Model**

The measurement model is based on the idea that if we want to perform a division, such as $6 \div 2$, it is the same as asking 'how many twos are in six?' One helpful way to convey the meaning of a model such as this is the use of a 'storyline' question which uses language and context to convey the model. For example, a storyline for $\frac{1}{2} \div \frac{1}{3}$ could consist of the questions:

- How many $\frac{1}{3}$ pounds of candy are in $\frac{1}{2}$ pounds of candy?
- A race is $\frac{1}{2}$ miles long. A lap is $\frac{1}{3}$ of a mile. How many laps do runners have to run in this race?

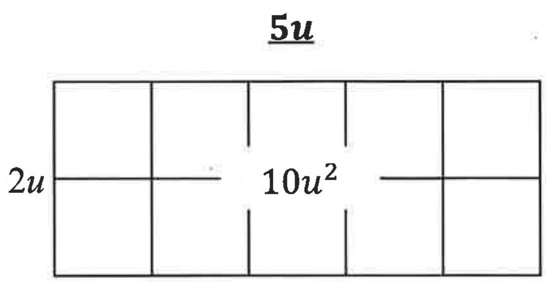
**Partitive Model**

The partitive model is based on the mathematical consequence that if $\frac{1}{2} \div \frac{1}{3} = x$, then it must be the case that $\frac{1}{3} \cdot x = \frac{1}{2}$. The following examples elaborate on the partitive model with storylines having similar content to the previous measurement model storylines:

- One-third of the weight of a box of candy weighs one-half of a pound. How much does the entire box of candy weigh?
- $\frac{1}{3}$ of the length of a race is $\frac{1}{2}$ of a mile. How long is the entire race?

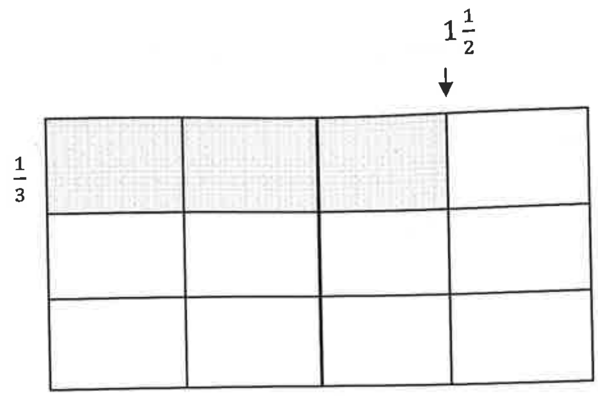
**Factors and Product (Area) Model**

Another use of the area model is to compute a division, $\frac{A}{w} = \ell$, which consists in forming a 'leading edge', $w$, of the rectangle and extending the edge until the given area $A$ has been accumulated, in which case the answer is the *top edge*, as in the following example for calculating $\frac{10}{2}$, we begin with a 'leading edge' of 2 $u$ and proceed to extend the edge out until we reach a total area of 10 $u^2$, in which case we see the *answer* in bold and underlined on the top edge.




Similar to how we did division with numbers, we will now give an example of how to teach division with fractions using the area model:

*Example 2.4.* $\frac{1}{2} \div \frac{1}{3}$




To understand how the area model works for fraction division, the denominator $\frac{1}{3}$ is used as a leading edge for a rectangle that will be constructed to make an area of $\frac{1}{2}$. To do this in the above example, the side edges are subdivided into three parts. To accumulate an area of $\frac{1}{2}$ of a unit square, vertical half marks need to be made, which creates a common denominator of $\frac{1}{6}$ subdivisions of the unit square. When these common denominator subdivisions are made, it can be seen that three of the small $\frac{1}{3} \times \frac{1}{2}$ squares form half of the unit square. The area being constructed, though, must extend out horizontally, hence the three $\frac{1}{6}$ squares extend out $1\frac{1}{2}$ units of length relative to the 'top edge'. This points to one of the key elements of the model, to not confuse area with edge. The answer when computing with the area model is the 'top edge'.