5.2 Divisibility Tests

divisibility test

If someone tells you that "the students in the class are wearing a total of 47 shoes" you immediately know that something is peculiar. You are alerted because this statement contrasts with the fact that all even numbers end with the digits 0, 2, 4, 6 or 8. That familiar fact is one example of a "divisibility test". It allows us to quickly spot which numbers — even very large numbers — are divisible by 2. In this section we will develop tests for divisibility by $2, 3, 4, 5, 8, 9, 10$, and 11.

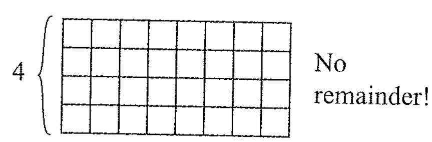
In the remainder of this chapter, we will often use letters $A, B, \ldots, k, l, \ldots$ and $a, b, \ldots$ to represent whole numbers. At any time you may assign them specific values (like $A = 30, k = 5$) to aid your understanding. For clarity, multiplication will be denoted by either a dot (as in $3 \cdot 5$) or no symbol (as in $3A$), rather than by the symbol $\times$ used in early elementary school.

We launch the topic with a clear definition. In this case it is especially important to be clear because there are at least *four* phrases commonly used for this one concept.

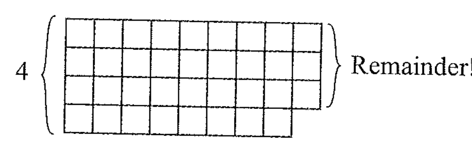
DEFINITION 2.1. *We say "A* ***is divisible by*** *k" whenever A is a multiple of k, that is, if $A = k \cdot a$ for some whole number a. The following phrases all have the same meaning:*

- *A is divisible by k*
- *A is a multiple of k*
- *k divides A*
- *k is a factor of A*

Divisibility can be illustrated by rectangular arrays. A number A is divisible by 4 if A squares can be arranged in a rectangular array with 4 rows of the same length.



32 is divisible by 4.



35 is *not* divisible by 4.

One also sometimes hears the phrase "k goes into A evenly" used as an informal expression meaning "if we divide A by k we get a whole number, with no remainder". In fact, that procedure — divide and see whether the remainder is zero — is the most direct way of checking divisibility, and the one students begin with.

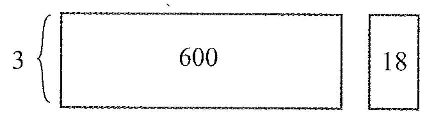
EXAMPLE 2.2. *Is* 537 *divisible by* 7*?*

The long division at right has a remainder of 5, so 537 is not divisible by 7.

$$
\begin{array}{r|l}
 & 76 \quad \text{R } 5 \\
7 & 537 \\
 & \underline{49} \\
 & 47 \\
 & \underline{42} \\
 & 5
\end{array}
$$

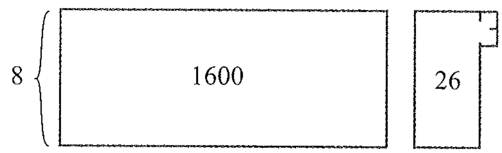
One can also use Mental Math methods.

EXAMPLE 2.3. *Is 618 divisible by 3?*



600 is a multiple of 3.
18 is a multiple of 3.
⇒ 618 is also.

EXAMPLE 2.4. *Is 1626 divisible by 8?*



1600 is a multiple of 8.
26 is not a multiple of 8.
⇒ 1626 isn't either.

Examples 2.3 and 2.4 use the same strategy: separate the given number into a large part which is clearly divisible by the given number, and a "leftover" part. Then check whether the leftover part is also divisible. We will repeatedly use that strategy to answer questions about divisibility. For that purpose we state the strategy as a lemma.

LEMMA 2.5 (Divisibility Lemma). *Suppose A is a number divisible by k. Then B is divisible by $k \iff A + B$ is divisible by k.*

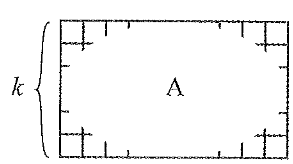
EXAMPLE 2.6. *Taking $A = 1600$, the Divisibility Lemma says that*

$$1626 \textit{ is divisible by } 8 \;\Leftrightarrow\; 26 \textit{ is divisible by } 8.$$

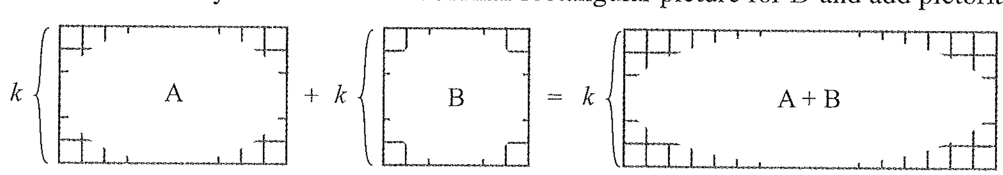
That is exactly the reasoning we used in Example 2.4. It reduces a hard problem to an easy one.

The double arrow ⇔ is read "if and only if". It means the statements to its left and right are equivalent, that is, if one is true, both are true. Thus the lemma is making two statements. The first is "if two numbers are divisible by k, then so is their sum". The second is the other way around: "if the sum of two whole numbers is divisible by k and one of the addends is also, then the other addend is also divisible by k." To prove Lemma 2.5 we must prove both statements.

Picture proof. Since A is divisible by k we can assemble A squares in a rectangular array with k squares in each column.

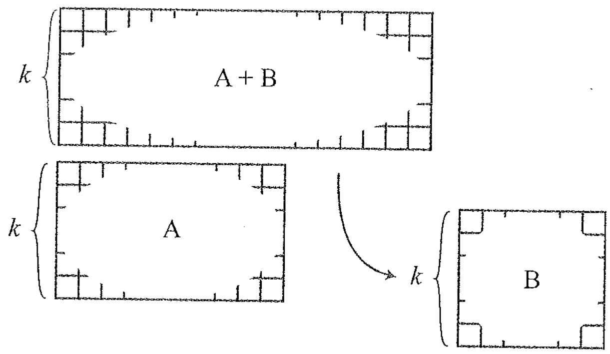


If B is also divisible by k we can draw a similar rectangular picture for B and add pictorially.



Thus whenever B is divisible by k then $A + B$ is also divisible by k.

Now for the other way around ("the converse"). If A and $A + B$ are divisible by k, we can arrange our rectangular arrays in a comparison model picture.



This shows that B is also a multiple of k. □

The above pictures should seem familiar: they are rectangular arrays illustrating two versions of the Distributive Property (cf. Section 1.5). Apparently the Distributive Property is the key idea in both parts of the proof! With that in mind, we can convert the picture proof into an algebraic proof by (i) introducing additional letters to label the (horizontal) lengths of the rectangles and (ii) using the Distributive Property.

Algebraic Proof. Because A is divisible by k we can write $A = ka$ for some whole number a (cf. the first picture above). If B is also divisible by k, then $B = kb$ for some whole number b. By the Distributive Property, $A + B = ka + kb = k(a + b)$, so $A + B$ is divisible by k.

To go the other way, suppose that $A+B$ is divisible by k. Then $A+B = kc$ for some whole number c. Again using the Distributive Property, we have $B = (A + B) - A = kc - ka = k(c - a)$, which shows that B is divisible by k. □

We are now ready to state and prove some specific divisibility tests. These are all derived and proved in essentially the same way, using the method of Examples 2.3 and 2.4. Given a number n and a test number t, we use expanded form to write n as a sum of a large number which is a multiple of t and a remainder. The Divisibility Lemma then tells us that n is divisible by t if and only if the remainder is. Checking the divisibility of the remainder is relatively easy, and that becomes the test.

THEOREM 2.7 (Divisibility Tests for 2, 4, 5, 8, 10). *A number is divisible*

- *by* 10 *if and only if its last digit is* 0,
- *by* 5 *if and only if its last digit is* 0 *or* 5,
- *by* 2 *if and only if its last digit is* $0, 2, 4, 6,$ *or* 8,
- *by* 4 *if and only if its last two digits are a number divisible by 4,*
- *by* 8 *if and only if its last three digits are a number divisible by 8.*

EXAMPLE 2.8.

a) $32,424$ *is even and is divisible by* 4 *because* 24 *(the last two digits) is divisible by* 4.

b) $98,234,484$ *is divisible by* 4 *because* 84 *is a multiple of* 4 *(since* 80 *is a multiple of* 4*). However, this number is not divisible by* 8 *since* $484 = 480 + 4$ *is not a multiple of* 8.

c) $1,943,537,208$ *is divisible by* 8 *because* $208 = 200 + 8$ *is a multiple of* 8.

The following two paragraphs show how divisibility tests are proved. Read these very carefully; they can be used as models for the proofs of other divisibility tests that you will do for homework. The idea of the first proof is to write the given number as a multiple of 10 plus its last digit. This is always possible by expanded form. For example, $4837 = 4000 + 800 + 30 + 7 = 4830 + 7 = 10 \cdot 483 + 7$. The second proof similarly uses expanded form to write the given number as a multiple of 100 plus its last two digits, as in $3762 = 3700 + 62 = 100 \cdot 37 + 62$.

Proof of the Divisibility Test for 2. Using expanded form, any whole number N can be written

$$N = 10a + b$$

where b is the last digit. But $10a = 2(5a)$ is divisible by 2, so the Divisibility Lemma (using $10a$ for A) says that N is a divisible by 2 if and only if b is divisible by 2, i.e., $b = 0, 2, 4, 6, 8$. □

Proof of the Divisibility Test for 4. Using expanded form we can write any whole number N as

$$N = 100a + b$$

where b is the last two digits. But $100a = 4(25a)$ is divisible by 4 so, by the Divisibility Lemma, N is divisible by 4 if and only if b is. □

EXERCISE 2.9. *Write down a proof for the Divisibility Test for 5 by duplicating the above proof of the Divisibility Test for 2 and modifying the last sentence appropriately.*

THEOREM 2.10 (Divisibility Tests for 3 and 9). *A number is divisible*
- *by 3 if and only if the sum of the digits is divisible by 3,*
- *by 9 if and only if the sum of the digits is divisible by 9.*

EXAMPLE 2.11. *a) The number 378 is divisible by both 3 and 9 because the sum of the digits $3 + 7 + 8 = 18$ is divisible by 3 and 9.*
b) The number 3822 is divisible by 3 but not by 9. Why?

Proof of the Divisibility Test for 3 and 9. Consider a three digit number N with digits abc (the proof with more digits is similar). In expanded form

$$N = 100a + 10b + c.$$

Note that $100a = 99a + a$ and $10b = 9b + b$, we can rewrite this as

$$N = (99a + 9b) + (a + b + c).$$

Now $(99a + 9b) = 9(11a + b)$ is divisible by both 3 and 9. By the Divisibility Lemma, N is divisible by 3 (or 9) if and only if $a + b + c$ is divisible by 3 (or 9). But $a + b + c$ is the sum of the digits! □

"Casting out nines." Here is a simple trick that makes checking divisibility by 3 and 9 very fast and easy: When checking divisibility by 9, it is not actually necessary to find sum of the digits; we need only determine whether the sum is a multiple of 9. Thus we can ignore any digits that are 9, and pairs of digits that sum to 9. For example, to check 429761, we can "cast out" the 9 and the $2 + 7$; that leaves $4 + 6 + 1 = 11$ so this number is not divisible by 9. Similarly, when checking divisibility by 3 one can "cast out threes".

THEOREM 2.12 (Divisibility Test for 11). *A number is divisible by 11 if and only if the number formed by*

$$(\textit{sum of odd-position digits}) - (\textit{sum of even-position digits})$$

is a multiple (positive, negative, or zero) of 11.

For example, the number 82,819 is divisible by 11 since $(8+8+9)-(2+1)=22$ is a multiple of 11. Similarly, the number 123 is not divisible by 11 since $(1+3)-(2)=2$ is not a multiple of 11. Other examples:

$$\begin{array}{rcl} 3894 & \longrightarrow & (8+4)-(3+9)=0 \implies 3894 \text{ is divisible by } 11, \\ 7491 & \longrightarrow & 5-16=-11 \implies 7491 \text{ is divisible by } 11, \\ 618,392 & \longrightarrow & 6-23=-17 \implies 618{,}392 \text{ is not divisible by } 11. \end{array}$$

Proof of the Divisibility Test for 11. We only give the proof for a four digit number N. Writing N in expanded form and rearranging,

$$\begin{aligned} N &= 1000a+100b+10c+d \\ &= (1001a+99b+11c)+(-a+b-c+d). \end{aligned}$$

But $1001a+99b+11c = 11(91a+9b+c)$ is divisible by 11. By the Divisibility Lemma, 11 divides N if and only if 11 divides $-a+b-c+d$. But that is exactly the difference $(b+d)-(a+c)$ between the odd-position and the even-position digits of N. □