Subtraction with Number Lines

Many measurement situations, such as time and distance, naturally lend themselves to a number line representation. Let's use an example to explore this model. Joanne had 8 feet of rope and used 3 feet to stake a tent. How much rope does she have left? Try to represent this problem with a number line and then read on. . . .

As with addition, we can use a number line to solve the problem in either of two ways (see Figure 3.9).

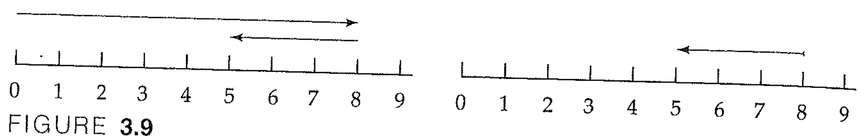


FIGURE 3.9

Do both ways make sense? Does one way feel more comfortable to you? As with the addition number lines, the left diagram more closely resembles the physical representation of the problem.

Properties of Subtraction

Think back to the properties of addition—identity, commutativity, associativity, and closure. Do those same properties hold for subtraction? Think and then read on. . . .

Some students think that subtraction has an identity property, because if we take away zero from a number, its value does not change; that is, $a - 0 = a$. This is true; however, if we reverse the order, the result is not true—that is, $0 - a \neq a$. Therefore, we generally say that the operation of subtraction does not have an identity property. In higher mathematics, we say that the operation of subtraction has a right-identity but not a left-identity. After examining a few cases, you can see that the operation of subtraction does not possess the commutative property or the associative property. The commutative property is not immediately understood by children. I recall one first grader arguing that $3 - 5$ was 0 because "you can't have less than nothing" and another arguing that it was 2 because "you just turn them around." Finally, the operation of subtraction is not closed for the set of natural numbers because the difference of two numbers can be a negative number.

INVESTIGATION 3.2 How Far Is It?

This problem, which actually happened to me, provides a good opportunity for you to apply your developing problem-solving tools and your understanding of subtraction and number line models to a real problem. I had to go to a meeting one day in Concord, New Hampshire. It was a business meeting, so I was to be reimbursed for my mileage. When I left Keene, I noted that my odometer read 26,688. Unfortunately, I forgot to check my odometer in Concord. However, on the way home, I saw a sign that said I was 36 miles from Keene. At that point, my odometer said 26,768. I now had enough information to determine the round-trip distance between Keene and Concord. What did I come up with? Try to solve this problem on your own before reading on. If you get stuck, what problem-solving tools might be useful?

DISCUSSION

Of course, I could have waited until I had gotten home, taken the mileage then, and subtracted 26,688 from that number. However, that would not be a challenging problem, and I did this actual problem because I was aware that there was a reasonable probability that I would forget to note my mileage when I got home!

This problem practically begs for a number line diagram. However, as many of my students have told me, creating good diagrams is easier said than done. To make this point more concrete, many students tell me that when they see a good diagram, it often makes sense, but that they don't know how to create the diagram themselves. In this problem, a useful diagram often emerges in two stages. For example, students respond in different ways to the diagram in Figure 3.10. Some students don't immediately understand the diagram, others understand it but don't find that it leads to the answer, whereas still others see it and "Eureka!" the solution to the problem literally appears right before their eyes. If you find that this diagram is not a eureka experience, what might be added to the diagram or how it could be modified? Please try to modify it yourself before reading on. . . .

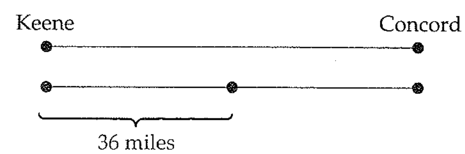


FIGURE **3.10**

There are different ways in which students find this diagram useful. For some students, the diagram points to a further step in the problem. For example, they now know what to do with 26,688 and 26,768. For other students, the diagram tells them that they have enough information to find the round-trip distance. Yet other students use algebraic notation and see two equations. Depending on how you see the problem, your modified diagram might look like either of the ones in Figure 3.11. Using either of these diagrams, or one of your own making, can you solve the problem now? Try to do so before reading on. . . .

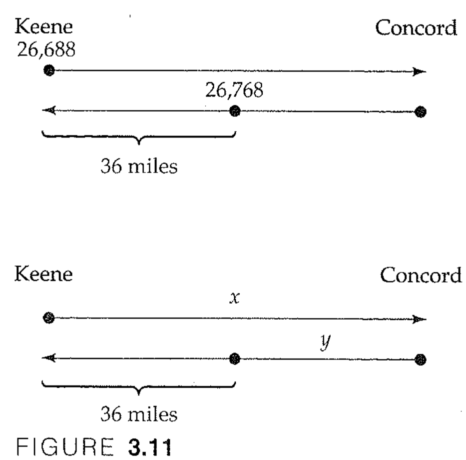


FIGURE **3.11**

One key to this problem is to use given information to produce more information. If we subtract 26,688 from 26,768, we get 80; what does that tell us? It tells us that I had traveled 80 miles at that point; adding this 80 to 36, we know the round-trip distance. A student using algebra would have an equation, $x + y = 80$, and would come to the same realization: The round-trip distance is 116 miles.

BEYOND THE CLASSROOM

In Investigation 3.2, we found the mathematical solution to the problem. Had I remembered to check my odometer in Concord or in Keene, however, it very well might not have shown that I drove 58 miles to Concord or 116 miles round trip. Do you see why? Think and then read on. . . .

The 36-mile sign does not mean that I am 36 miles away from my house. It means that I am 36 miles from Keene. But that begs the question "36 miles from *where* in Keene?" For example, think of a sign telling you that you are 36 miles from New York City. What does that mean? Does it mean 36 miles to the center of New York City or to the city limits? In general, it means that you are 36 miles from what the map makers consider to be the geographical center of the city, not the city limits. Do you see why mileage signs are made this way? How they determine the center is another question. What do you think is the center of the town closest to you?

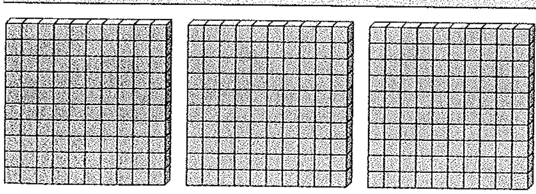
Understanding Subtraction in Base 10

Let us now examine some standard and nonstandard algorithms for subtraction. Again, recall the Alphabitian problems; for a child, subtracting 32 − 14 can be as overwhelming and confusing as many students originally found DA − BC in the Alphabitian system.

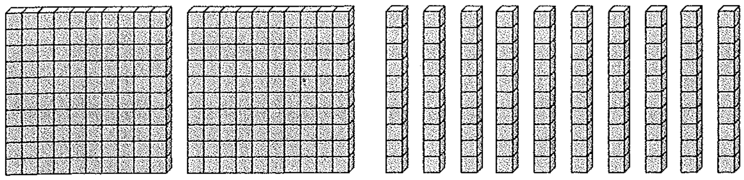
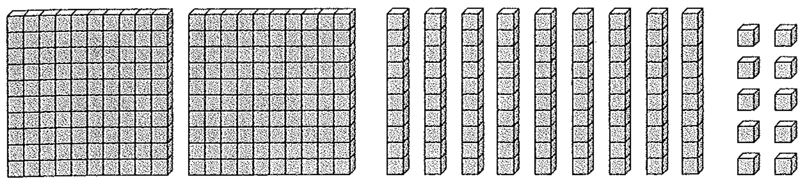
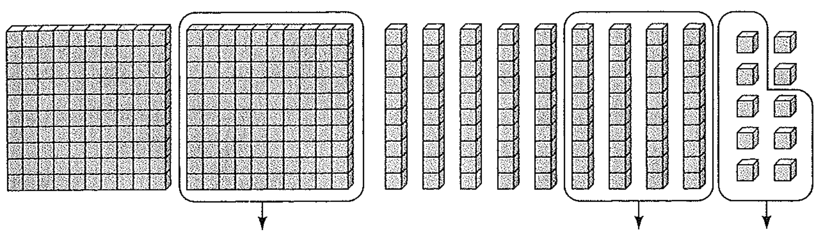
> When nothing remains, put down a small circle so that the place be not empty, but the circle must occupy it, so that the number of places will not be diminished when the place is empty and the second be mistaken for the first.[3]

This quote comes from a text written by Al-Khowarizmi in A.D. 825 as he tried to explain the procedure for subtracting in the new (Hindu) numeration system. Do you understand what he is saying? What function does the circle serve?

Researchers have found that when subtraction problems have zeros, the success rate for most third graders goes down drastically. However, this need not be so. Let us examine the following problem: 300 − 148. Although there is no single procedure that all students use to solve this problem (unless they are forced to by their teacher), we will examine here how this problem might be solved going from left to right.

Using manipulatives	Using words	Using symbols
	We need to regroup 300 so that we can "take away" 148.	$\begin{array}{r} 300 \\ -\,148 \\ \hline \end{array}$

[3] Karl Menninger, *Number Words and Number Symbols: A Cultural History of Numbers* (Cambridge, Mass.: MIT Press, 1969), p. 413.

Using manipulatives	Using words	Using symbols
	We can trade one of the 3 hundreds for 10 tens, giving us 2 hundreds and 10 tens, which has the same value as 3 hundreds.	2 1 $\not{3}$ 0 0 − 1 4 8
	Next, we can trade one of the tens for 10 ones. We now have 2 hundreds, 9 tens, and 10 ones, which still has the same value as 3 hundreds.	2 9 1 $\not{3}\not{0}$ 0 − 1 4 8
	Now we can take away 1 hundred, 4 tens, and 8 ones.	2 9 1 $\not{3}\not{0}$ 0 − 1 4 8 1 5 2

Justifying the Standard Algorithm

It is beyond the scope of this book to prove formally every algorithm, procedure, and theorem. What is essential is that elementary students and teachers understand the algorithms that they use—that they understand the whys of the algorithm. Does the explanation above help you to understand the left-to-right subtraction procedure? If not, please read it again or refer to your notes from the Alphabitia explorations, or work with a friend so that it does make sense.

Some students find that representing the problem in expanded form is helpful. A key to understanding this algorithm is to understand the equivalence of 3 hundreds and 2 hundreds, 9 tens, and 10 ones. Although they look different, and the numerals are different (300 versus $29^{1}0$), the amounts are equal. Another key to understanding this algorithm is to know why we needed to do the regrouping. This is easier to see at the physical level; without these regroupings, we cannot literally "take away" 1 flat, 4 longs, and 8 units.

3 hundreds + 0 tens + 0 ones →	2 hundreds + 10 tens + 0 ones →	2 hundreds + 9 tens + 10 ones
−1 hundred + 4 tens + 8 ones	−1 hundred + 4 tens + 8 ones	−1 hundred + 4 tens + 8 ones
Cannot subtract the tens or the ones	Cannot subtract the ones	Subtraction possible

An Alternative Algorithm

It is important to note that the standard algorithm (in the discussion above) is more closely connected to the take-away context for subtraction than to the comparison or missing addend context. For example, consider the problem

LEARNING

It is important to reemphasize that "standard" algorithms are not the "right" ones, or even the "best" ones, but rather the ones that, for various reasons, have become most widespread. Many educators believe that more harm than good is done by forcing all students to learn the "standard" algorithms.

802 − 238 in a missing addend context: Joan wants to buy a computer for $802; if she has $238, how much more does she need? If we are solving this problem in the missing addend context, we are asking what number combined with 238 gives us 802. One possible algorithm would begin with the smaller amount and add to it, beginning with the hundreds place and finishing with the ones place. Both of my children invented this procedure when I asked them to tell me how they determined the difference before they were taught the standard algorithm.

WHAT IS THOUGHT	WHAT IS WRITTEN	
	"Answer" column	"Working" column
How many hundreds can we add to 238 without going over 802?	500	738
We can add 500, which brings us to 738.		
Now, how many tens can we add to 738 without going over 802?	60	798
We can add 60, which brings us to 798.		
How many ones do we need to add to 798 to get to 802?	4	802
We need 4 ones.		

The answer of 564 is obtained by adding the numbers in the answer column. Though many college students find this algorithm initially awkward, when elementary children are encouraged to develop their own means for answering subtraction problems, we see algorithms like this one much more often than the more standard ones that many textbooks contain. Exploration 3.5 illustrates the fact that the standard algorithm is one of many possible algorithms and is not necessarily the one that children find easiest to understand.

As with addition, the composition and decomposition of numbers enables us to understand better both the standard subtraction algorithms and other algorithms for subtraction. When you explored subtraction in Alphabitia in Exploration 3.3, you may have found that not all students subtracted in the same way; some went left to right, some went right to left, and for some it depended on whether there were zeros in the subtrahend. Regardless of the procedure, subtracting with confidence requires that we be comfortable with the idea of decomposing and recomposing the number.

"Carrying" and "Borrowing"

Traditionally, elementary teachers have tended to use the word *carry* when regrouping for addition and *borrow* when regrouping for subtraction. Many current elementary teachers do not use these words. Can you guess why? Think and then read on. . . .

One problem is that the term *borrow* is misleading. When I borrow an egg from a neighbor or when I borrow a tissue, that is different from "borrowing" a ten.

Another problem with carry and borrow is that the terms imply two different processes, when in fact the same process is involved: trading (regrouping). We trade up when carrying, and we trade down when borrowing. For example, consider the related addition and subtraction problems 36 + 28 = 64 and $64 - 36 = 28$. When we add the digits in the ones place, we trade in 10 of the ones for 1 ten, and so 14 ones becomes 1 ten and 4 ones. In the subtraction problem, we go the other way: We trade a ten for 10 ones, and so 1 ten and 4 ones become 14 ones.

(14 ones → 1 ten and 4 ones) (1 ten and 4 ones → 14 ones)

$$\begin{array}{r} {}^{1} \\ 36 \\ +\,28 \\ \hline 64 \end{array} \qquad\qquad \begin{array}{r} {}^{5\,1} \\ \not{6}\,4 \\ -\,3\,6 \\ \hline 2\,8 \end{array}$$

Many adults do not realize that these processes are virtually identical; they have never stopped to think about it. When learning the algorithms for the first time, young children find it much easier if there is a reason for what they do. The connection between these two processes is further examined in Exploration 3.3.

Summary

In this section, we have examined the operations of addition and subtraction. You now know that addition can mean to combine or join two sets; it can also mean to increase a set by a certain amount. You know that subtraction can mean to take away an amount from a set, it can mean to ask how much bigger one set is than another, and it can also mean to ask how much a set must increase in order to get to a certain amount. Especially in the case of subtraction, you have seen that these meanings are not always obvious to children.

Addition and subtraction problems arise from discrete or measurement (continuous) contexts. We can represent these problems in various ways: with circles, with part-whole rectangles, or with number lines. The choice of model sometimes depends on personal preference and sometimes depends on the nature of the problem being solved.

These representations help us to see connections between addition and subtraction. In one sense, addition consists of adding two parts to make a whole. In one sense, subtraction consists of having a whole and a part and needing to find the value of the other part. You are becoming more comfortable with the idea of composition and decomposition, which appears repeatedly throughout mathematics. In one sense, addition is an act of composition, putting together two parts to make a new whole. In one sense, subtraction is an act of decomposition, breaking a whole into a given part and a part we need to find.

We have examined patterns in the addition table. Recognizing these patterns helps children to become more comfortable with adding. We have seen that describing the patterns that we see requires some thinking and that mathematical vocabulary and definitions, developed over the centuries, can make communication easier and reduce ambiguity.

You have learned that there are certain properties that hold for addition of whole numbers but not for subtraction: identity, commutativity, associativity, and closure. The identity property may seem obvious now, but it will be seen as an important part of the whole set of properties in Section 3.2.

Finally, we have examined both standard and alternative algorithms for whole-number addition and subtraction. You have learned that the algorithms we currently use in the United States are not the only possible procedures for computing. The ability not only to compute with these algorithms but also to explain why they work is a crucial part of developing mathematical power.

In Section 3.2, we will do a similar examination of multiplication and division, and then we will examine the relationships among all four operations.