CHAPTER 9

Decimals, Rational and Real Numbers

What exactly is "the set of numbers"? We actually have two different answers to that question. Ever since Section 1.1 we have used the number line model in which numbers are points on the number line. That geometric viewpoint makes the ordering, relative size, and closeness of numbers apparent.

It is then reasonable to say that *every* point on the line specifies a number. The numbers which arise in this way are called "real numbers".

DEFINITION 1. *A real number is a point on the number line.*

But numbers are more than just geometric points — they can be added, subtracted, multiplied, and divided. In previous chapters we introduced those operations in the order they are developed in elementary school, defining them first for whole numbers and then extending them first to fractions and then to negative numbers. The resulting set of numbers are called "rational numbers".

DEFINITION 2. *A rational number is a positive or negative fraction, i.e., a number that can be written as $\frac{a}{b}$ where a and b are integers with $b \neq 0$.*

At this point we know how to add, subtract, multiply, and divide rational numbers:

$$\frac{3}{4} - \frac{1}{2} = \frac{1}{4}, \qquad \frac{4}{9} \div \frac{2}{3} = \frac{2}{3}, \qquad \text{and} \qquad \frac{3}{5} \times \left(-\frac{4}{7}\right) = -\frac{12}{35}.$$

Thus we have two distinct descriptions of numbers: a geometric one describing real numbers as points on the number line, and an "algebraic" one describing the four operations on rational numbers. In this chapter we will explore the interesting story of how these two approaches are related, and how both are related to decimals.

9.1 Decimals

Decimals specify points on the number line by repeatedly subdividing intervals into tenths ("deci" means tenth). Just as a mailing address locates someone by specifying a state, a city in that state, a street in that city and a house on that street, a decimal number gives the "address" of a point on the number line — the digits give successively more accurate information which, together, precisely locate a single point.

decimals

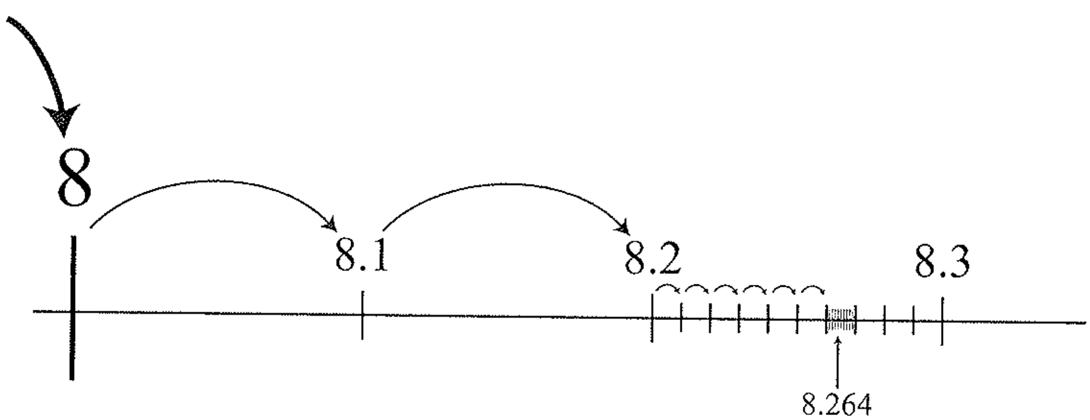
This process is an extension of place value notation. Whole numbers are written as a string of digits that specify multiples of the "denominations" $1, 10, 100, 1000$, etc. according to their position. *Decimals* are numbers written with the same logical notation, but including place value positions corresponding to the denominations $\frac{1}{10}, \frac{1}{100}$, etc. obtained by dividing 1 by powers of 10. Thus each decimal has an expanded form, such as

$$8.264 = \left(8 \times 1\right) + \left(2 \times \frac{1}{10}\right) + \left(6 \times \frac{1}{100}\right) + \left(4 \times \frac{1}{1000}\right).$$

decimal point

Notice the role of the decimal point: it identifies the ones digit (the digit immediately to its left). That determines the place values of all the digits and allows us to distinguish the numbers 8.264, 82.64, 826.4 and 8264. Because a decimal point at the beginning of a number is hard to spot, one usually inserts an initial 0, for example writing 0.37 instead of .37.

The decimal 8.264 can be thought of as directions for walking along the number line starting at 0: take 8 steps of size 1, then 2 steps of size $\frac{1}{10}$, 6 steps of size $\frac{1}{100}$, and 4 steps of size $\frac{1}{1000}$. Because the increments shrink rapidly, only a few decimal places are needed to locate a point on the number line with great precision. The system is remarkably simple and efficient.



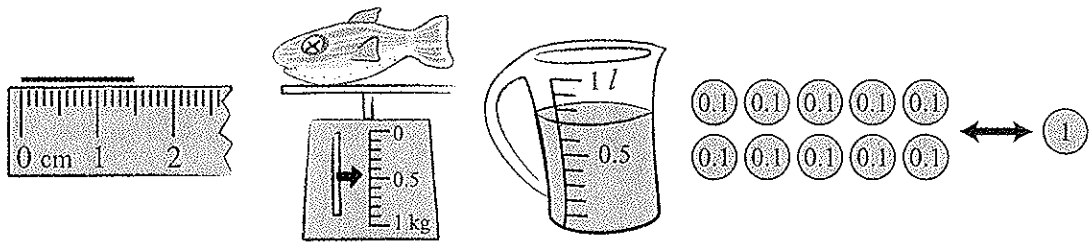
Like other place value ideas, decimals are more difficult for children than one might think. The historical record also suggests that the idea is not obvious — decimals and the decimal point were not introduced until the year 1610! Clearly, teaching decimals requires a renewed emphasis on place value notions.

The Primary Mathematics curriculum introduces decimals in the fourth grade. At that point students are familiar with addition and subtraction of fractions with like denominators (but have not yet seen multiplication and division of fractions). The students' knowledge of fractions is used to launch the topic of decimal numbers. Decimals are introduced as a new way to write tenths:

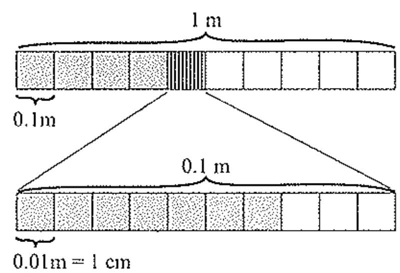
$$\frac{8}{10}\text{ cm} \qquad \text{and} \qquad 0.8\text{ cm}$$

are two ways of writing the same number. Similarly the fraction $\frac{67}{100}$ can be written as 0.67.

Decimals can be illustrated using measurement models with metric measurements and set models using chips and coins (a penny is \$0.01 and a dime is \$0.1 dollars).



The central idea behind decimals — that we repeatedly subdivide into ten equal parts — can be illustrated by a measurement model

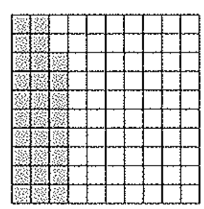


$0.4 + 0.07 = 0.47$

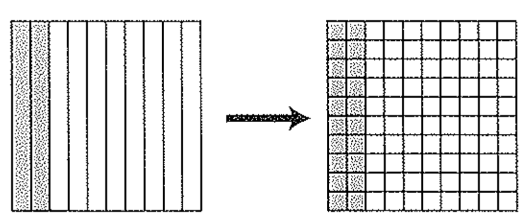
hundreds square

and by the area model called the *hundreds square*. Each column or row of the hundreds square represents 0.1 and each small square represents 0.01, and the appropriate area is shaded.

0.28



Students sometimes mistakenly depict 0.2 by shading two small squares of a hundreds square. That error can be exposed and corrected by first illustrating 2 tenths and then overlaying a hundreds square.



0.2 = 2 tenths   0.20 = 20 hundredths

EXERCISE 1.1. *Here are some problems that help students convert between fractions and decimals and understand the use of place value concepts for decimals. (Fill in the blanks as you read.)*

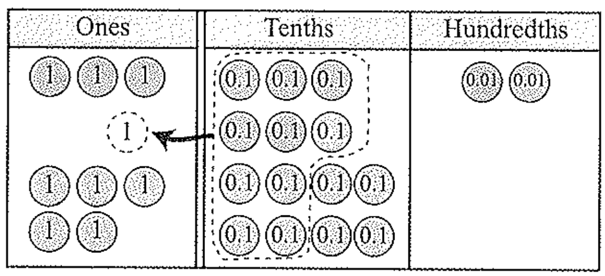
- 0.2 *cm is* $\frac{2}{\square}$ *of 1 centimeter.*
- 12 *tenths* $=$ 1 *and* 2 *tenths* $= 1.2 = 1 + 0.2$.
- $0.01 = \frac{1}{100} =$ *1 hundredth.*
- $0.37 = \frac{3}{10} + \frac{7}{\square} = 0.3 + \underline{\qquad}$.
- $\$0.43 = 43$ *cents* $= \frac{43}{\square}$ *of a dollar* $=$ *4 dimes and 3 pennies.*

Operations with Decimals

The basic place value principle of whole number addition — "add the ones, tens, and hundreds separately, regrouping when necessary" — carries over to decimals. Chip models make that clear.

EXAMPLE 1.2. *Find* $3.62 + 5.8$.

$$\begin{array}{r} {\scriptstyle 1} \\ 3.62 \\ +\ 5.8 \\ \hline 9.42 \end{array}$$

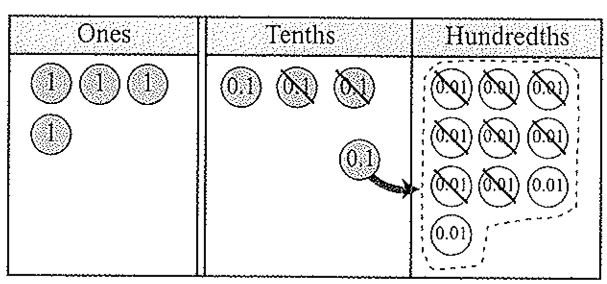


Notice that aligning the decimal point automatically aligns the ones, tenths, and hundredths columns. Once that is understood, the chip model is no longer needed.

EXAMPLE 1.3. *Find* $4.4 - 2.28$.

In the chip model, this subtraction involves decomposing one tenth into 10 hundredths. To clarify that in the algorithm it is helpful to rewrite 4.4 as 4.40. The problem is then the same as 440 − 228 when we count in hundredths.

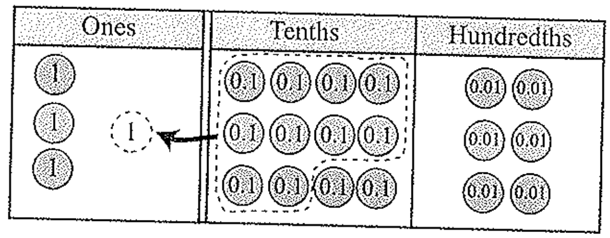
$$\begin{array}{r} {\scriptstyle 3\ 10} \\ 4.\not{4}\not{0} \\ -\,2.28 \\ \hline 2.12 \end{array}$$



We can multiply a decimal by a whole number by a similar approach: convert a whole number multiplication problem by counting in tenths, hundredths, or thousandths. Multiplication by a small whole number can again be illustrated by a chip model.

EXAMPLE 1.4. *Find* 1.42×3.

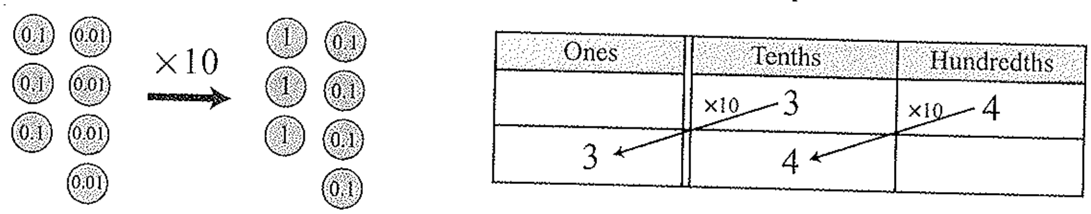
$$\begin{array}{r} {\scriptstyle 1} \\ 1.42 \\ \times \quad 3 \\ \hline 4.26 \end{array}$$



The two approaches to long division described in Section 3.4 also carry over to decimal division (we will give examples below). All four algorithms for decimals are covered at the beginning of Primary Math 4B; that placement provides an opportunity for students to review the algorithms in a new context.

Multiplying and Dividing by 10

One fact about decimals is especially important: one can multiply and divide by 10 simply by shifting the decimal point. This is another fact about place value which *must be taught explicitly*. The idea can be introduced using money and chip models. Given a pile of pennies and dimes, we can create a pile of coins worth ten times as much by replacing each penny with a dime, and each dime with a dollar. When that replacement is recorded in a place-value chart it appears as a shift to the left, or equivalently a shift of the decimal point.



$0.34 \times 10 = 3.4$

Similar pictures show that one divides by 10 by shifting the decimal point one place to the left.

This shift in the decimal point can also be explained using fractions:

$$0.234 \times 10 = \frac{234}{1000} \times 10 = \frac{234}{100} = 2.34.$$

EXERCISE 1.5. *Calculate* $5.7 \div 100$ *by converting to fractions, dividing, and converting back to decimals.*

shifting the decimal point

These examples demonstrate the very useful *shifting procedure:*

- To multiply a decimal number by 10, shift the decimal point 1 place to the right. To multiply by 100, shift the decimal point 2 places to the right, etc.
- To divide by 10, 100, etc. , similarly shift the decimal point to the left.

Multi-digit decimal multiplication and division

Teaching students how to multiply and divide decimals does not require returning to first principles and using chip models. It is much more efficient to build on students' knowledge of whole number and fraction arithmetic. In fact, it is easy to multiply decimals by converting to fractions, multiplying the fractions, and converting back to decimal notation. For example,

$$2.17 \times 3.4 = \frac{\square}{100} \times \frac{34}{10} = \frac{217 \times 34}{1000} = \frac{7378}{1000} = 7.378.$$

In the numerators of this calculation we see the whole number multiplication $217 \times 34 = 7378$. The denominators show that the number of decimal places in the answer (namely 3) is the sum of the number of decimal places in 1.02 and 2.3 (2 in the first, 1 in the second). That observation leads to a general procedure for decimal multiplication.

1. Erase the decimal points and multiply the factors as if they were whole numbers.
2. Insert a decimal point in the product so that the total number of decimal places is the same on the two sides of the equation.

There are two alternative ways to locate the decimal point.

2'. By estimating. In the example above, the estimate $2.17 \times 3.4 \approx 2 \times 3 = 6$ shows where to insert the decimal point in the digits 7378.

2". By shifting decimal points. Using compensation we can multiply one factor by 10 or 100 and divide the other factor by the same amount without changing the answer. This is done by shifting the decimal point in opposite directions on the two factors.

$$0.08 \times 206 = 8 \times 2.06 \approx 8 \times 2 = 16$$

The same principles apply to division. Compensation now says that the shifts should be in the *same* direction; that allows us to convert any decimal division to a division problem involving only whole numbers. For example,

$$162.800 \div 0.037 = 162800 \div 37.$$

This is obvious in fraction form: multiplying numerator and denominator by 1000 shows that

$$162.8 \div 0.037 = \frac{162.8}{0.037} = \frac{162800}{37}.$$

EXAMPLE 1.6. *Estimate* $0.52 \div 2.9$, *then find the value to 2 decimal places.*

Rounding, we see that $0.52 \div 2.9 \approx 0.5 \div 3$, which is between 0.1 and 0.2. For better accuracy, we can convert $0.52 \div 2.9$ to the whole number division $5.2 \div 29$ and use long division:

$$
\begin{array}{r|l}
 & 0.179 \\ \hline
29 & 5.2 \\
 & -2.9 \\ \hline
 & 2.30 \\
 & -2.03 \\ \hline
 & 270 \\
 & -261 \\ \hline
 & 9
\end{array}
\quad \Longrightarrow \quad \boxed{0.52 \div 2.9 \approx 0.18}
$$

Such examples lead to a general procedure for dividing a decimal by another decimal:

> 1. Shift the decimal point of the divisor to make it a whole number and then shift the decimal point of the dividend the same number of places.
> 2. Find the quotient by long division, aligning the decimal points of the quotient and the dividend.

Homework Set 37

1. Write each of the following in decimal form.
   a) $4 \times 10 + 3 \times 1 + 2 \times \frac{1}{10} + 8 \times \frac{1}{100}$
   b) $7 \times \left(\frac{1}{10}\right)^2 + 3 + 5 \times \left(\frac{1}{10}\right)^7$
   c) $18 \times \left(\frac{1}{10}\right)^5$
   d) 1.7×1.7
   e) 1.6×1.8
   f) $12 \times 10{,}010$
   g) $32 \div 16000$
   h) $52{,}000{,}000 \div 130{,}000$
   i) 0.032×0.0010001
2. *(Mental Math)* Calculate mentally and show how you did it.
   a) $17.32 - 9.97$
   b) $3.2 \times 5.3 + 1.1 \times 3.2$
   c) $3.7 + 33.8 + 6.3$
3. *(Mental Math)* Estimate using Simple Estimation (see Section 3.5).
   a) $48.2 \div 2.8$
   b) 0.21×125.2
   c) $13{,}345 \div 652$
   d) 0.49×0.0057
   e) $0.035 \div 0.0068$
   f) 14.2×16.3