Elementary and Middle School Mathematics

Teaching Developmentally

FIFTH EDITION

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CHAPTER 18

Developing Concepts of Ratio and Proportion

Proportional reasoning has been referred to as the capstone of the elementary curriculum and the cornerstone of algebra and beyond (Lesh, Post, & Behr, 1987). The ability to reason proportionally was a hallmark of Piaget's distinction between concrete levels of thought and formal operational thought. It represents the ability to begin to understand multiplicative relationships where most arithmetic concepts are additive in nature.

Big Ideas

1. A ratio is a comparison of any two quantities. A key developmental milestone is the ability of a student to begin to think of a ratio as a distinct entity, different from the two measures that made it up.
2. Proportions involve multiplicative rather than additive comparisons. Equal ratios result from multiplication or division, not from addition or subtraction.
3. Proportional thinking is developed through activities involving comparing and determining the equivalence of ratios and solving proportions in a wide variety of problem-based contexts and situations without recourse to rules or formulas.

Content Connections

Proportional reasoning is indeed the cornerstone of a wide variety of topics in the middle and high school curriculum.

- **Fractions** (Chapter 15): Equivalent fractions are found through a multiplicative process; numerators and denominators are multiplied or divided by the same number. Equivalent ratios can be found in the same manner. In fact, part-whole relationships (fractions) are an example of ratio.
- **Similarity** (Chapter 20): When two figures are the same shape but different sizes (i.e., similar), they constitute a visual example of a proportion. The ratios of linear measures in one figure will be equal to the corresponding ratios in the other.
- **Data Graphs** (Chapter 21): A relative frequency histogram shows the frequencies of different related events compared to all outcomes (visual part-to-whole ratios). A box-and-whisker plot shows the relative distribution of data along a number line and can be used to compare distributions of populations of very different sizes.
- **Probability** (Chapter 21): A probability is a ratio that compares the number of outcomes in an event to the total possible outcomes. Proportional reasoning helps students understand these ratios, especially in comparing large and small sample sizes.
- **Algebra** (Chapters 23 and 24): Much of algebra concerns a study of change and, hence, rates of change (ratios) are particularly important. In this chapter you will see that the graphs of equivalent ratios are straight lines passing through the origin. The slope of the line is the unit ratio. Slope itself is a rate of change and is an important component in understanding algebraic representations of related quantities.

Proportional Reasoning

So that the reading of this chapter is based on common ground, several exercises are provided for your exploration. Each typifies the relationships that make up the concepts of ratio and proportion. It does not matter what your personal mathematical acquaintance with ratio actually is. Try each of the following problems. You may wish to discuss and explore the exercises in a group or with a friend.

LAPS

Yesterday, Mary counted the number of laps she ran and recorded the amount of time it took. Today she ran fewer laps in more time than yesterday. Did she run faster, slower, or at about the same speed today—or can't you tell? What if she had run more laps in more time?

SIMILAR SHAPES

Place a piece of paper over the dot grid in Figure 18.1. Using the dots as a guide, draw a shape that is *like* the one shown but larger. How many different shapes that are like the given shape can you draw and still stay within the grid provided?

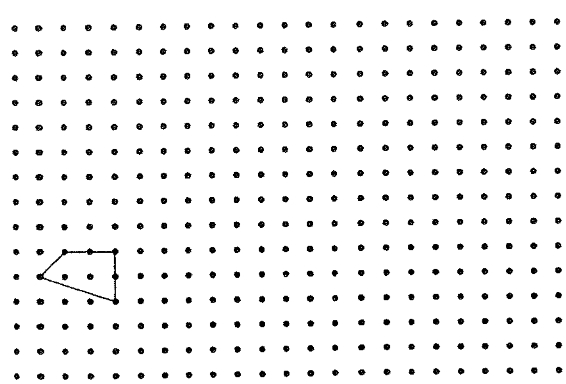


FIGURE 18.1 Draw some shapes just like this, only larger.

TOMATOES

At the farmers' market, Jones sells tomatoes at 5 pounds for $7 whereas Smith sells his tomatoes at 4 pounds for $6. Which farmer sells his tomatoes at a cheaper price?

SHORT AND TALL

Mr. Green sells trophies on the basis of height. A short trophy measures 6 paper clips tall, as shown in Figure 18.2. According to his price chart, this short trophy sells for $4.00. Mr. Green has a taller trophy that is worth $6.00 according to the same scale. How many paper clips tall is the $6.00 trophy?

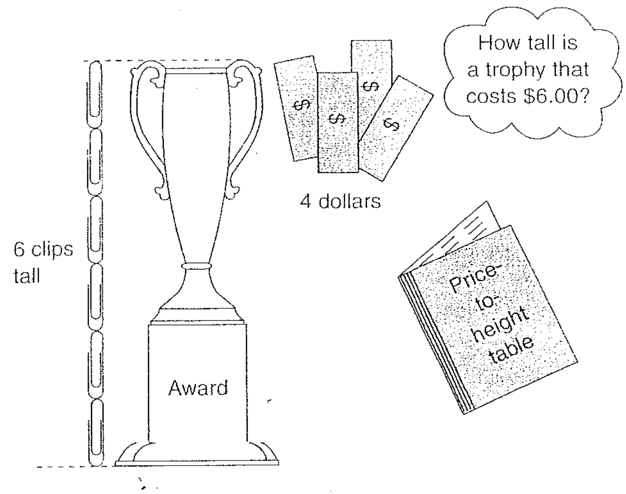


FIGURE 18.2 The "Short and Tall" trophy problem.

Examples of Ratios in Different Contexts

A *ratio* is an ordered pair of numbers or measurements that expresses a comparison between the numbers or measures. To students, ratios in different settings or contexts may present very different ideas and different difficulties.

Part-to-Whole Ratios

Ratios can express comparisons of a part to a whole, for example, the ratio of girls to all students in the class. Because fractions are also part-whole ratios, it follows that every fraction is also a ratio. In the same way, percentages are ratios, and in fact, percentages are sometimes used to express ratios. Probabilities are ratios of a part of the sample space to the whole sample space.

Part-to-Part Ratios

A ratio can also express one part of a whole to another part of the same whole. For example, the number of girls in the class can be compared to the number of boys. For other examples, consider Democrats to Republicans or peanuts to cashews. Although the probability of an event is a part-to-whole ratio, the *odds* of an event happening is a ratio of the number of ways an event can happen to the number of ways it cannot happen—a part-to-part ratio.

Rates as Ratios

Both part-to-whole and part-to-part ratios compare two measures of the same type of thing. A ratio can also be a *rate*. A rate is a comparison of the measures of two different things or quantities; the measuring unit is different for each value.

In the "Short and Tall" exercise, the comparison between paper clips and money is a ratio that is a rate—in this case, dollars per paper clip. All prices are rates and are, therefore, ratios: 69 cents each, 3 for a dollar, 12 ounces for $1.39.

The "Laps" exercise involves comparisons of time to distance, another example of rate. In that example the reasoning is qualitative rather than quantitative, but the ratio concept is basically the same. All rates of speed are comparisons of time and distance: for example, driving at 55 miles per hour (time to distance) or jogging at 9 minutes per mile (distance to time).

Miles per gallon, square yards of coverage per gallon of paint, passengers per busload, and roses per bouquet are all rates. Relationships between two units of measure are also rates or ratios, for example, inches per foot, milliliters per liter, and centimeters per inch.

Other Examples of Ratio

In geometry, the ratios of corresponding parts of similar geometric figures are always the same, as in the "Similar Shapes" exercise. The diagonal of a square is always $\sqrt{2}$ times a side. π (pi) is the ratio of the circumference of a circle to the diameter. The trigonometric functions can be developed from ratios of sides of right triangles. The slope of a line or of a roof is a ratio of rise for each unit of horizontal distance or run.

In nature, the ratio known as the *golden ratio* is found in many spirals, from nautilus shells to the swirls of a pinecone or a pineapple. Artists and architects have used the same ratio in creating shapes that are naturally pleasing to the eye.

Proportions

As students begin to reflect on a variety of examples of ratios, they will also begin to see two or more comparisons that are in the *same ratio*. In the "Similar Shapes" exercise, the ratio of any two sides of a small shape is the same as the ratio of the corresponding sides of a larger but similar shape. The "Tomatoes" problem also involved two ratios. Here the task was to compare the ratios to decide which was greater. The comparison of ratios forms an important category of tasks for students in developing proportional reasoning.

A *proportion* is a statement of equality between two ratios. Different notations for proportions can be used, for example:

$$3 : 9 = 4 : 12 \quad \text{or} \quad \frac{3}{9} = \frac{4}{12}$$

These might be read "3 is to 9 as 4 is to 12" or "3 and 9 are in the same ratio as 4 and 12."

A ratio that is a rate usually includes the units of measure when written, for example:

$$\frac{\$12.50}{1 \text{ gallon}} = \frac{\$37.50}{3 \text{ gallons}}$$

In the "Short and Tall" problem, you completed a proportion, or "solved" it, by completing the second ratio. Finding one number in a proportion when given the other three is called *solving* a proportion.

Proportional Reasoning and Children

Proportional reasoning is difficult to define in a simple sentence or two. Proportional reasoning is not something that you either can or cannot do. It is, as the example problems show, both a qualitative and quantitative process. According to Lamon (1999), the following are a few of the characteristics of proportional thinkers:

- Proportional thinkers have a sense of covariation. That is, they understand relationships in which two quantities vary together and are able to see how the variation in one coincides with the variation in another.
- Proportional thinkers recognize proportional relationships as distinct from nonproportional relationships in real-world contexts.
- Proportional thinkers develop a wide variety of strategies for solving proportions or comparing ratios, most of which are based on informal strategies rather than prescribed algorithms.
- Proportional thinkers understand ratios as distinct entities representing a relationship different from the quantities they compare.

It is estimated that more than half of the adult population cannot be viewed as proportional thinkers (Lamon, 1999). That means that we do not acquire the habits and skills of proportional reasoning simply by getting older. On the other hand, Lamon's research and that of others indicate that instruction can have an effect, especially if rules and algorithms for fraction computation, for comparing ratios, and for solving proportions are delayed. Students may need as much as three years' worth of opportunities to reason in multiplicative situations in order to adequately develop proportional reasoning skills. Premature use of rules encourages students to apply rules without thinking and, thus, the ability to reason proportionally often does not develop.

Additive Versus Multiplicative Situations

Consider the following problem adapted from the book *Adding It Up* (National Research Council, 2001).

Two weeks ago, two flowers were measured at 8 inches and 12 inches, respectively. Today they are 11 inches and 15 inches tall. Did the 8-inch or 12-inch flower grow more?

STOP Before reading further, find and defend two different answers to this problem.

One answer is that they both grew the same amount—3 inches. This correct response is based on additive reasoning. That is, a single quantity was added to each measure to result in the two new measures. A second way to look at the problem is to compare the amount of growth to the original height of the flower. The first flower grew $\frac{3}{8}$ of its height while the second grew $\frac{3}{12}$. Based on this multiplicative view ($\frac{3}{8}$ *times as much* more), the first flower grew more. This is a proportional view of this change situation. An ability to understand the difference between these situations is an indication of proportional reasoning.

Considerable research has been conducted to determine how children reason in various proportionality tasks and to determine if developmental or instructional factors are related to proportional reasoning (for example, see Karplus, Pulos, & Stage, 1983; Lamon, 1993; Lo & Watanabe, 1997; Noelting, 1980; and Post, Behr, & Lesh, 1988).

The research provides direction for how to help children develop proportional thought processes. Some of these ideas are outlined here.

1. Provide ratio and proportion tasks in a wide range of contexts. These might include situations involving measurements, prices, geometric and other visual contexts, and rates of all sorts.
2. Encourage discussion and experimentation in predicting and comparing ratios. Help children distinguish between proportional and nonproportional comparisons by providing examples of each and discussing the differences.
3. Help children relate proportional reasoning to existing processes. The concept of unit fractions is very similar to unit rates. Research indicates that the use of a unit rate for comparing ratios and solving proportions is the most common approach among junior high students even when cross-product methods have been taught. (This approach is explained later.)
4. Recognize that symbolic or mechanical methods, such as the cross-product algorithm, for solving proportions do not develop proportional reasoning and should not be introduced until students have had many experiences with intuitive and conceptual methods.

NCTM Standards In 1989, the *Curriculum Standards* noted that proportional reasoning "was of such great importance that it merits whatever time and effort must be expended to assure its careful development" (NCTM, 1989, p. 82). The emphasis on proportional reasoning is similarly reflected in the 2000 *Standards*. The *Principles and Standards* authors have focused on the need for an integrative approach, one that involves "percent, similarity, scaling, linear equations, slope, relative frequency histograms, and probability" (NCTM, 2000, p. 212).

Informal Activities to Develop Proportional Reasoning

Four categories of informal activities are suggested here: selection of equivalent ratios, comparison of ratios, scaling with ratio tables, and construction and measurement activities. Each provides a different opportunity for the development of proportional reasoning. The activities are not in any definitive sequence, nor are they designed to teach specific methods for solving proportions.

Equivalent-Ratio Selections

In selection activities, a ratio is presented, and students select an equivalent ratio from others presented. The focus should be on an intuitive rationale for why the pairs selected are in the same ratio. Sometimes numeric values will play a part to help students develop numeric methods to explain their reasoning. In later activities, students will be asked to construct an equivalent ratio without choices being provided.

It is extremely useful in these activities to include pairs of ratios that are not proportional but have a common difference. For example, $\frac{5}{8}$ and $\frac{9}{12}$ are not equivalent ratios, but the corresponding differences are the same: $8 - 5 = 12 - 9$. Students who focus on this additive relationship are not seeing the multiplicative relationship of proportionality.

ACTIVITY 18.1

Look-Alike Rectangles

Draw a collection of 13 rectangles on one sheet of paper. Use a sheet of centimeter grid paper, and trace the rectangles using whole-number measures for at least one dimension of each rectangle. Make four rectangles with sides in the ratio of 1 to 4, four in the ratio 2 to 5, four in the ratio 5 to 8, and one square about midsize to the other 12 rectangles. Label the rectangles A to M in a mixed order.

The task is to group the rectangles into three sets of four that "look alike" with one "oddball." When they have decided on their groupings, stop and discuss reasons they classified the rectangles as they did. Be prepared for some students to try to match sides or look for rectangles that have the same amount of difference between them. Do not evaluate any rationale offered. Next have them measure and record the sides of each rectangle to the nearest half-centimeter and calculate the ratios of the short to long sides for each. You may want to prepare a little worksheet as shown in Figure 18.3. Discuss these results, and ask students to offer explanations. If the groups are formed of proportional (similar) rectangles, the ratios within each group will all be the same.

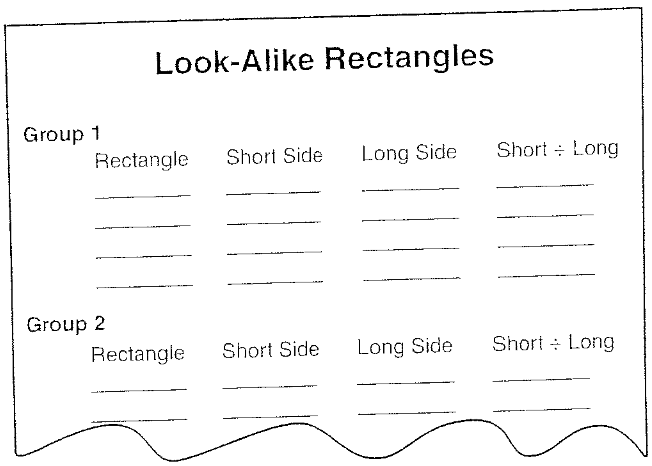


FIGURE 18.3 **A recording sheet for "Look-Alike Rectangles."**

From a geometric standpoint, "Look-Alike Rectangles" is an activity about similarity. The two concepts—proportionality and similarity—are closely connected. However, if the activity were done with figures other than rectangles, the congruent angles of similar figures would defeat the purpose.

Another feature of proportional rectangles can be observed by stacking like rectangles aligned at one corner, as in Figure 18.4. Place a straightedge across the diagonals, and you will see that opposite corners also line up. If the rectangles are placed on a coordinate axis with the common corner at the origin, the slope of the line joining the corners is the ratio of the sides. Here is a connection between proportional reasoning and algebra.

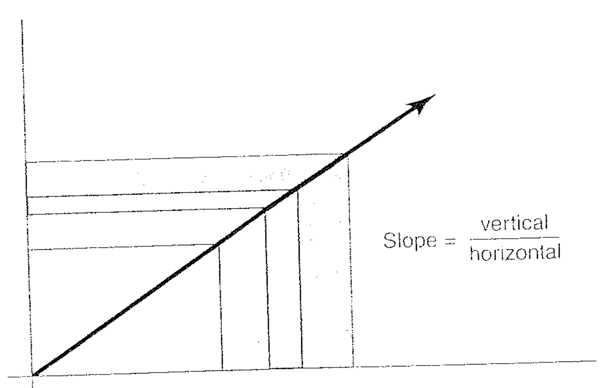
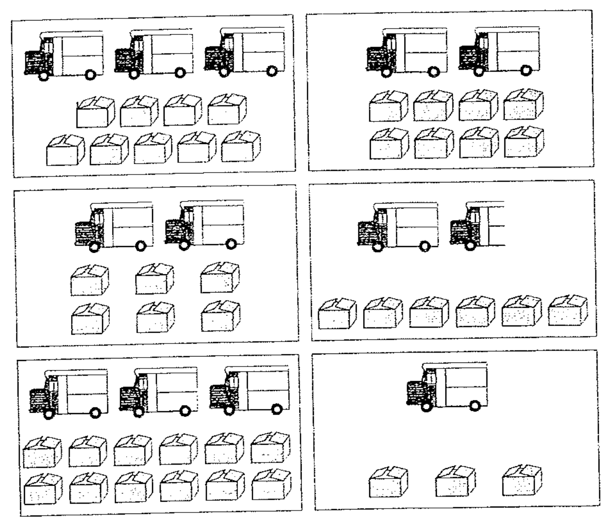


FIGURE 18.4 **The slope of a line through a stack of proportional rectangles is equal to the ratio of the two sides.**

ACTIVITY 18.2

Different Objects, Same Ratios

Prepare cards with distinctly different objects, as shown in Figure 18.5. Given one card, students are to select a card on which the ratio of the two types of objects is the same. This task moves students to a numeric approach rather than a visual one and introduces the notion of ratios as rates. A unit rate is depicted on a card that shows exactly one of either of the two types of objects. For example, the card with three boxes and one truck provides one unit rate. A unit rate for the other ratio is not shown. What would it be? Objects paired with coins or bills is a way to introduce price as a ratio.



On which cards is the ratio of trucks to boxes the same? Also, compare trucks to trucks and boxes to boxes.

FIGURE 18.5 **Rate cards: Match cards with the same rate of boxes per truck.**

Comparing Ratios

An understanding of proportional situations includes being able to distinguish between two ratios as well as to identify those ratios that are equivalent. Consider the following problem:

Two camps of Scouts are having pizza parties. The Bear Camp ordered enough so that every 3 campers will have 2 pizzas. The leader of the Raccoons ordered enough so that there would be 3 pizzas for every 5 campers. Did the Bear campers or the Raccoon campers have more pizza to eat?

STOP Get some scrap paper and solve this problem without using any numeric algorithms such as cross-products. You may want to draw pictures or use counters, but there is no prescribed method to use.

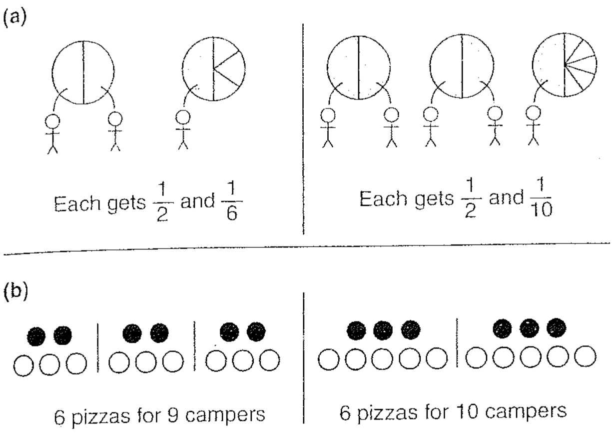


FIGURE 18.6 **Two informal methods for comparing two ratios.**

Figure 18.6 shows two different possibilities for informal methods.

When the pizzas are sliced up into fractional parts (Figure 18.6a), the approach is to look for a unit rate—pizzas per camper. A sharing approach has been used for each ratio just as described for fractions in Chapter 15. But notice that this problem does not say that the camps have only 3 and 5 campers, respectively. Any multiples of 2 to 3 and 3 to 5 can be used to make the appropriate comparison. This is the approach used in Figure 18.6b. Three "clones" of the 2-to-3 ratio and two clones of the 3-to-5 ratio are made so that the number of campers getting a like number of pizzas can be compared. From a vantage of fractions, this is like getting common numerators. Because there are more campers in the Raccoon ratio (larger denominator), there is less pizza for each camper.

The following activity suggests some similar comparison tasks.

ACTIVITY 18.3

Comparing Ratios

Pose problems to students that are similar to the following. Allow them to solve the problems in any manner they wish as long as they can explain why their answers make sense. Do not allow any algorithms that the students cannot defend in the context of the problem.

a. Terry can run 4 laps in 12 minutes. Susan can run 2 laps in 5 minutes. Who is the faster runner?
b. Jack and Jill were picking strawberries at the Pick Your Own Berry Patch. Jack "sampled" 5 berries every 25 minutes. Jill ate 3 berries every 10 minutes. If they both pick at about the same speed, who will bring home more berries?
c. Some of the hens in Farmer Brown's chicken farm lay brown eggs and the others lay white eggs. Farmer Brown noticed that in the large hen house he collected about 4 brown eggs for every 10 white ones. In the smaller hen house the ratio of brown to white was 1 to 3. In which hen house do the hens lay more brown eggs?
d. Talks-A-Lot phone company charges 70¢ for every 15 minutes. Reaching Out phone company charges \$1.00 for 20 minutes. Which company is offering the cheaper rate?
e. Which rectangle is "fatter": a 3 × 5 rectangle or an 8 × 14 rectangle?

The suggested problems in Activity 18.3 are simply to provide you with some ideas. You can easily make up your own. You can also change the numbers to make the tasks easier or harder.

Scaling with Ratio Tables

Ratio tables or charts that show how two variable quantities are related are often good ways to organize information. Consider the following table:

Acres	5	10	15	20	25		
Pine trees	75	150	225				

If the task were to find the number of trees for 65 acres of land or the number of acres needed for 750 trees, students can easily proceed by using addition. That is, they can add 5's along the top row and 75's along the bottom row until the problems are solved. Although this is efficient and orderly, it is an additive procedure and does little as a task to promote proportional reasoning. As illustrated in the next activity, the instructional "trick" is to select numbers that require some form of multiplicative thinking.

ACTIVITY 18.4

Using Ratio Tables

Given a situation like one of the following, the task is to build a ratio table and use it to answer the question. Tasks are adapted from Lamon (1999, p. 183).

a. A person who weighs 160 pounds on Earth will weigh 416 pounds on the planet Jupiter. How much will a person weigh on Jupiter who weighs 120 pounds on earth?
b. At the local college, 5 out of every 8 seniors live in apartments. How many of the 30 senior math majors are likely to live in an apartment?
c. The tax on a purchase of \$20 is \$1.12. How much tax will there be on a purchase of \$45.50?
d. When in Australia you can exchange \$4.50 in U.S. dollars for \$6 Australian. How much is \$17.50 Australian in U.S. dollars?

(a)

÷2 ÷2 ×3

Earth weight	160	80	40	120
Jupiter weight	416	208	104	312

(b)

add

Earth weight	160	80	40	120
Jupiter weight	416	208	104	312

(c)

add

÷8 ×5

Earth weight	160	20	100	120
Jupiter weight	416	52	260	312

FIGURE 18.7 160 pounds on Earth is 416 pounds on Jupiter. If something weighs 120 pounds on Earth, how many pounds would it weigh on Jupiter? Three solutions using ratio tables.

The tasks in this activity are typical "solve the proportion" tasks. One ratio and part of a second are given with the task being to find the fourth number. However, tasks such as these should come long before any formal approach is suggested. Further note that in no case is it easy to simply add or subtract to get to the desired entry. Rather, the student should use a ratio table to solve the problem. Figure 18.7 shows three different ways to solve the Jupiter weight task using ratio tables.

The format of these ratio tables is not at all important. Some students may not use a table format at all and simply draw arrows and explain in words how they got from one ratio to another. You may find value in a more structured format. The following problem and the table in Figure 18.8 are taken from Lamon (1999, p. 233). Notice that the numbers are not "nice" at all.

Cheese is $4.25 per pound. How much will 12.13 pounds cost?

	Pounds	Cost	Notes
A	1	4.25	Given
B	10	42.50	A × 10
C	2	8.50	A × 2
D	0.1	0.425	A ÷ 10
E	12.1	51.425	B + C + D
F	0.01	0.0425	D ÷ 10
G	0.03	0.1275	F × 3
H	12.13	51.5525	E + G

FIGURE 18.8 A more structured ratio table. The notes column shows what was done in each step. The task is to find the cost of 12.13 pounds.

The format in Figure 18.8 allows for easier tracing of what was done at each step. The format is just that—a format. It is not the same as an algorithm. For any problem there are likely to be several different reasonable ratio tables. In applying this technique, students are using multiplicative relationships to transform a given ratio into an equivalent ratio. As Lamon points out, the process is not at all random. Students should mentally devise a plan for getting from one number to another. Consider the following problem:

How many pounds of grass seed can be purchased for $18 if you can buy 28 pounds for $35?

STOP

Before reading further, write down a plan for moving from 35 to 18. Then create a ratio table using your plan to solve the problem. Compare your strategy with someone else's, or try to find another one yourself.

One possible plan for getting from 35 to 18 as follows: 35 ÷ 5 is 7; 7 × 2 is 14. (Now you need 4 more.) Go back to 7: 7 ÷ 7 is 1; 1 × 4 is 4. Now add 4 and 14 to make 18. When these same steps are applied to 28, the foregoing problem is solved.

The tasks suggested in Activity 18.4 have quite reasonable numbers. However, as you can see from the cheese example, it is quite possible to use this technique with almost any numbers. By using easy multiples and divisors, often the arithmetic can be done mentally.

Any ratio table, even a quite simple one such as you might find in textbooks, provides data that can be graphed. Make each axis correspond to one of the quantities in the table. This idea is developed in the next activity.

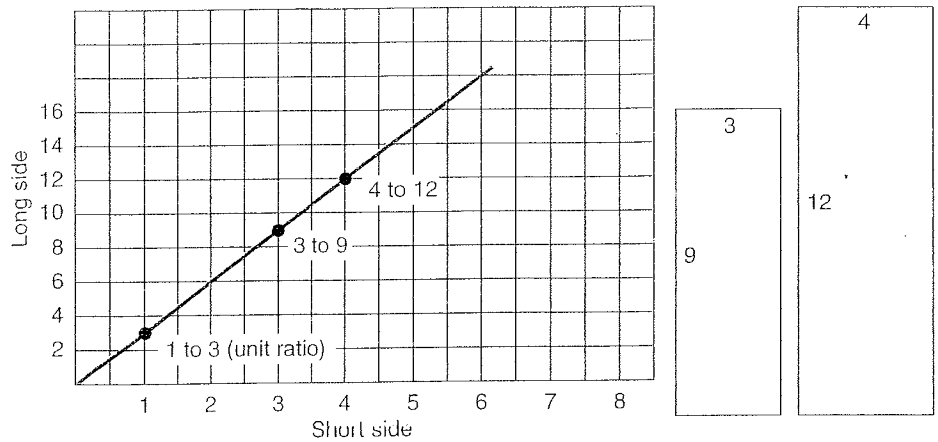


FIGURE 18.9 **Graphs show ratios of sides in similar rectangles.**

ACTIVITY 18.5

Graphs Showing Ratios

Have students make a graph of the data from a collection of equal ratios that they have scaled or discussed. The graph in Figure 18.9 is of the ratios of two sides of similar rectangles. If only a few ratios have actually been computed, the graph can be drawn carefully and then used to determine other equivalent ratios. This is especially interesting when there is a physical model to coincide with the ratio. In the rectangle example, students can draw rectangles with sides determined by the graphs and compare them to the original rectangles. A unit ratio can be found by locating the point on the line that is directly above or to the right of the number 1 on the graph. (There are actually two unit ratios for every ratio. Why?) Students can then use the unit ratio to scale up to other values and check to see that they are on the graph as well. Note that the slope of any line through the origin is a ratio.

Graphs provide another way of thinking about proportions, and they connect proportional thought to algebraic interpretations. All graphs of equivalent ratios fall along straight lines that pass through the origin. If the equation of one of these lines is written in the form $y = mx$, the slope m is always one of the equivalent ratios. Figure 18.10 shows the graph of the prices of widgets.

Construction and Measurement Activities

In these activities, students make measurements or construct physical or visual models of equivalent ratios in order to provide a tangible example of a proportion as well as look at numeric relationships.

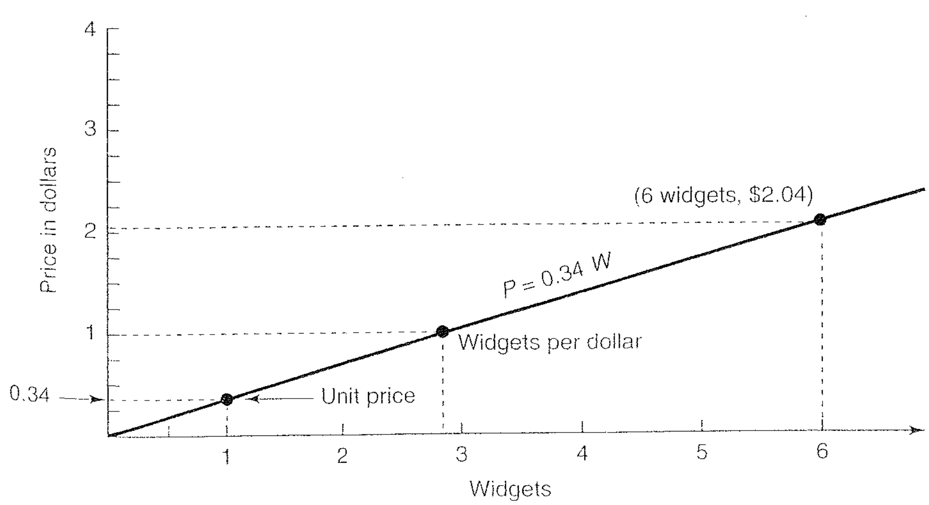


FIGURE 18.10 **Graph of price-to-item ratios.**

ACTIVITY 18.6

Different Units, Equal Ratios

Cut strips of adding machine tape all the same length, and give one strip to each group in your class. Each group is to measure the strip using a different unit. Possible units include different Cuisenaire rods, a piece of chalk, a pencil, the edge of a book or index card, or standard units such as inches or centimeters. When every group has measured the strip, ask for the measure of one of the groups, and display the unit of measure. Next, hold up the unit of measure used by another group, and have the class compare it with the first unit. See if the class can estimate the measurement that the second group found. The ratio of the measuring units should be the inverse of the measurements made with those units. For example, if two units are in a ratio of 2 to 3, the respective measures will be in a ratio of 3 to 2. Examine measurements made with other units. Finally, present a unit that no group has used, and see if the class can predict the measurement when made with that unit.

Activity 18.6 can be profitably extended by providing each group with an identical set of four strips of quite different lengths. Good lengths might be 20, 50, 80, and 120 cm. As before, each group measures the strips using a different unit.

This time, have each group enter data into a common spreadsheet. (Alternatively, share group data so that all groups can enter data on their own spreadsheets.) Figure 18.11 shows what a spreadsheet might look like for three groups. A template can be prepared ahead of time, or students can create their own spreadsheets. Almost all spreadsheets will offer a variety of graphing options. In this activity, bar graphs show the actual measurements for each group and circle graphs show each measure in ratio to the sum of the measures (i.e., a percentage of total measures.)

Once the graphs are completed, there are numerous opportunities to observe and explore proportions. The bar graphs, though different, all look "alike." Since the circle graphs illustrate the ratios rather than the actual measurements, they will be identical or nearly so. Within ratios (for a set of strips) and between ratios (one unit to another) are easily calculated with the spreadsheet. (Within and between ratios are discussed later in the chapter.)

Continue the exploration by introducing a new strip. If you know its measure with any one of the units, what will its measure be with the other units? Similarly, if a new unit of measure is introduced, how can the measures of the strips be determined? Can this be done by comparing the new unit with an old one? If a known strip is measured with the new unit, can all other measures and ratios be determined?

Bar graphs and circle plots are also easily made with a TI-73 graphing calculator. If technology is not available, this same activity can be done by hand. To make circle graphs, use the hundredths disk in the Blackline Masters.

The connection between proportional reasoning and the geometric concept of similarity is very important. Similar figures provide a visual representation of proportions, and proportional thinking enhances the understanding of similarity. Whenever similarity is discussed, ratios in the figures should almost certainly be explored. The next two activities are aimed at this connection.

FIGURE 18.11
A spreadsheet can be used to record data, create tables of interesting ratios, and produce bar and circle graphs.
Source: Screen reprinted with permission from Apple Computer, Inc.

Four strips measured with three different units.

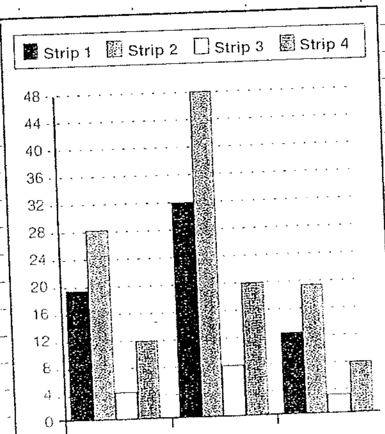
	Unit A	Unit B	Unit C
Strip 1	19.2	32	12.8
Strip 2	28.8	48	19.2
Strip 3	4.8	8	3.2
Strip 4	12.0	20	8.0

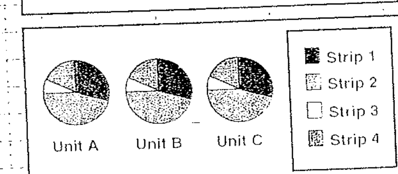
Within Ratios

1 to 2	0.7	0.7	0.7
1 to 3	4.0	4.0	4.0
1 to 4	1.6	1.6	1.6
2 to 3	6.0	6.0	6.0
2 to 4	2.4	2.4	2.4
3 to 4	0.4	0.4	0.4

Between Ratios

	A to B	A to C	B to C
Strip 1	0.6	1.5	2.5
Strip 2	0.6	1.5	2.5
Strip 3	0.6	1.5	2.5
Strip 4	0.6	1.5	2.5





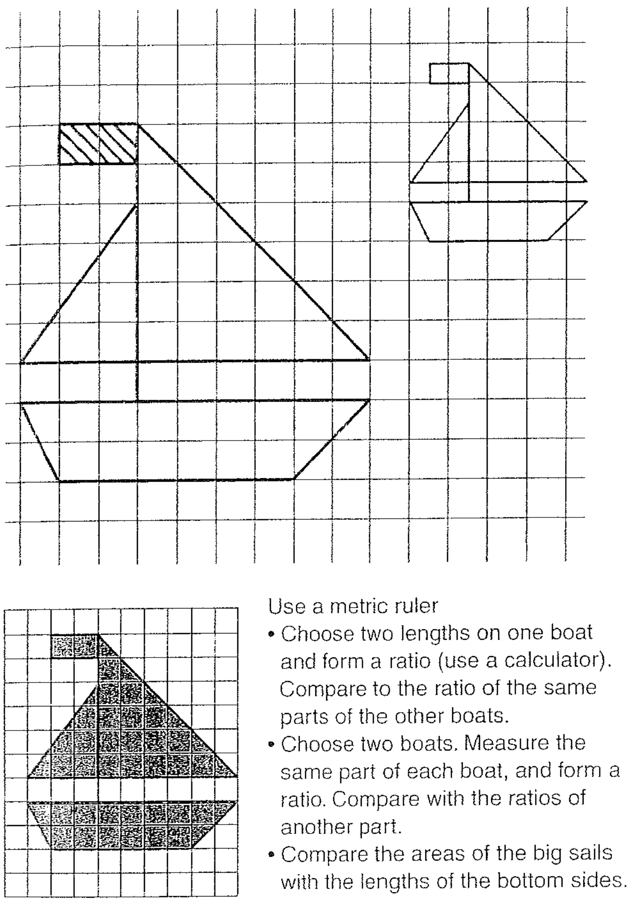


FIGURE 18.12 **Comparing similar figures drawn on grids.**

ACTIVITY 18.7

Scale Drawings

On grid or dot paper (see the Blackline Masters), have students draw a simple shape using straight lines with vertices on the dots. After one shape is complete, have them draw a larger or smaller shape that is the same as or similar to the first. This can be done on a grid of the same size or a different size, as shown in Figure 18.12. After completing two or three pictures of different sizes, the ratios of the lengths of different sides can be compared.

Corresponding sides from one figure to the next should all be in the same ratio. The ratio of two sides within one figure should be the same as the ratio of the corresponding two sides in another figure.

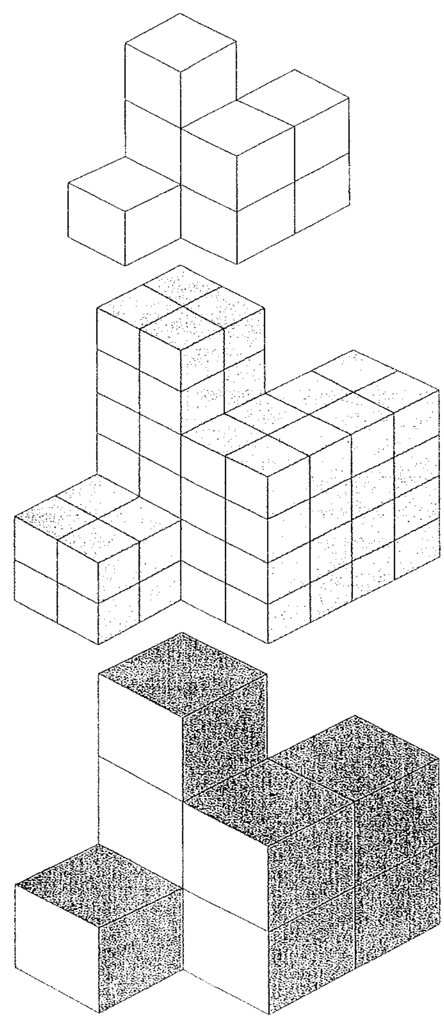
ACTIVITY 18.8

Length, Surface, and Volume Ratios

A three-dimensional version of Activity 18.7 can be done with blocks, as shown in Figure 18.13. Using 1-inch or 2-cm wooden cubes, make a simple "building." Then make a similar but larger building, and compare measures. A different size can also be made using different-sized blocks. To measure buildings made with different blocks, use a common unit such as centimeters.

Activities 18.7 and 18.8 involve area and volume as well as length. Comparisons of corresponding lengths, areas, and volumes in proportional figures lead to some interesting ratios. If two figures are proportional (similar), any two linear dimensions you measure will be in the same ratio on each, say, 1 to k. Corresponding areas, however, will be in the ratio of 1 to k^2, and corresponding volumes in the ratio of 1 to k^3. Try this with some constructions of your own.

As a means of contrasting proportional situations with additive ones, try starting with a figure on a grid or a building made with blocks and adding two units to every dimension in the figure. The result will be larger but will not look at all the same. Try this with a simple rectangle that is 1 cm by 15 cm. The new rectangle is twice as "thick" (2 cm) but



Similar "buildings" can be made by changing the number of blocks in each dimension (factor of change) or by using different-sized blocks.

FIGURE 18.13 **Similar constructions.**

MIDDLE GRADES

Connected Mathematics

Grade 7, *Comparing and Scaling*

Investigation 3: Comparing and Using Ratios

Context

This investigation occurs in the second week of the unit on ratio and proportions. In earlier activities, students explored ratios and percents to compare survey data from large populations with similar data gathered from their own class. Students used fractions, decimals, and percents to express ratios, and they compared ratios using their own strategies.

Task Description

The problem shown here is introduced in the context of students deciding on the best mix of juice for a camping trip.

Within the same investigation are two similar problems that are paraphrased here:

1a. A can of tomatoes will make sauce for 5 to 6 campers. How many cans should be purchased to make spaghetti for 240 campers? Five cans cost $4. How much will the cans of tomatoes cost?

2a. If pizzas are shared evenly, will a camper get more pizza sharing 4 pizzas with 10 campers or 3 pizzas with 8 campers?

2b. The ratio of 10-seat tables to 8-seat tables is 8 to 5. If there are just enough tables for all 240 campers, how many of each type are there?

Problem 3.1

Arvind and Mariah tested four juice mixes

Mix A	Mix B
2 cups concentrate 3 cups cold water	1 cup concentrate 4 cups cold water
Mix C	**Mix D**
4 cups concentrate 8 cups cold water	3 cups concentrate 5 cups cold water

A. Which recipe will make juice that is the most "orangey"? Explain your answer.

B. Which recipe will make juice that is the least "orangey"? Explain your answer.

C. Assume that each camper will get $\frac{1}{2}$ cup of juice. For each recipe, how much concentrate and how much water are needed to make juice for 240 campers? Explain your anwer.

The full unit contains six investigations, each with numerous real contexts. Scaling, the use of unit rates, and percentages are suggested techniques. However, no particular method is forced on the students.

only a bit longer. It will not appear to be the same shape as the original.

Dynamic geometry software such as *The Geometer's Sketchpad* (Key Curriculum Press) offers a very effective method of exploring the idea of ratio. In Figure 18.14, two lengths are drawn on a grid using the "snap-to-grid" option. The lengths are measured, and two ratios are computed. As the length of either line is changed, the measures and ratios are updated instantly. A screen similar to this could be used to discuss ratios of lengths as well as inverse ratios with your full class. In this example, notice that the second pair of lines has the same difference but that the ratios are not the same. A similar drawing could be prepared for the overhead on a transparency of a centimeter dot grid if software was not available.

The *Connected Mathematics* program places the main emphasis on proportional reasoning in the seventh grade. In the eighth grade, the proportional concepts are applied throughout the curriculum. The sample activity on this page is similar to the comparing ratio ideas you have read about earlier.

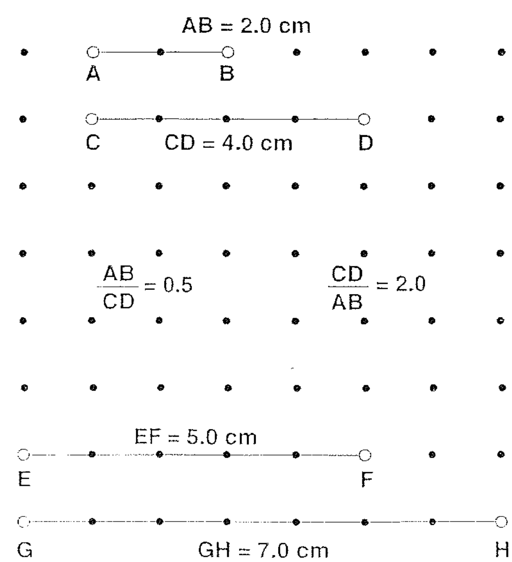


FIGURE 18.14 Dynamic geometry software or just a centimeter grid can be used to discuss ratios of two lengths.

NCTM Standards "Attention to developing flexibility in working with rational numbers contributes to students' understanding of, and facility with, proportionality. Facility with proportionality involves much more than setting two ratios equal and solving for a missing term. It involves recognizing quantities that are related proportionally and using numbers, tables, graphs, and equations to think about the quantities and their relationship" (p. 217).

Solving Proportions

The activities to this point have been designed to lead students to an intuitive concept of ratio and proportion to help in the development of proportional reasoning.

One practical value of proportional reasoning is to use observed proportions to find unknown values. Knowledge of one ratio can often be used to find a value in the other. Comparison pricing, using scales on maps, and solving percentage problems are just a few everyday instances where solving proportions is required. Students need to learn to set up proportions symbolically and to solve them.

Within and Between Ratios

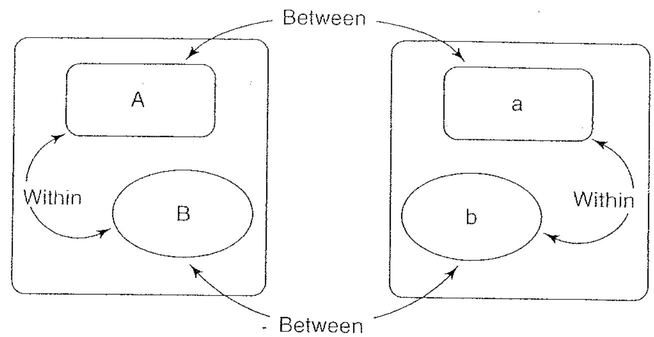
When examining two ratios, it is sometimes useful to think of them as being either *within* ratios or *between* ratios. A ratio of two measures in the same setting is a *within* ratio. For example, in the case of similar rectangles, the ratio of length to width for any one rectangle is a within ratio, that is, it is "within" the context of that rectangle. For all similar rectangles, corresponding within ratios will be equal.

A *between* ratio is a ratio of two corresponding measures in different situations. In the case of similar rectangles, the ratio of the length of one rectangle to the length of another is a between ratio; that is, it is "between" the two rectangles. For two similar rectangles, all of the between ratios will be equal. However, the between ratios for each pair of similar rectangles will be different.

STOP Consider three rectangles *A*, *B*, and *C*. *A* measures 2 × 6, *B* measures 3 × 9, and *C* measures 8 × 24. Find the within ratio for each rectangle. This should convince you that the rectangles are similar. Now examine the between ratios for *A* and *B*, and for *A* and *C*. Why are these ratios different?

Figure 18.5 (p. 302) shows six pictures of trucks and boxes. The within ratios are trucks to boxes (within one picture). The between ratios are from trucks to trucks and boxes to boxes. Be sure that you can distinguish within and between ratios in that figure.

The simple drawing in Figure 18.15 is a nice generic way of looking at two ratios and determining if a ratio is between



Within: $\frac{A}{B} = \frac{a}{b}$

Between: $\frac{A}{a} = \frac{B}{b}$

FIGURE 18.15 Given a proportional situation, the two between ratios and the two within ratios will be the same.

or within. A drawing similar to this will be very helpful to students in setting up proportions. Pick any two equivalent truck and box pictures and place the numbers in this figure.

An Informal Approach

Traditional textbooks show students how to set up an equation of two ratios involving an unknown, "cross-multiply," and solve for the unknown. This can be a very mechanical approach and will almost certainly lead to confusion and error. Although you may wish eventually to cover the cross-product algorithm, it is well worth the time for students to find ways to solve proportions using their own ideas first. If you have been exploring proportions informally, students will have a good foundation on which to build their own approaches.

To illustrate some intuitive approaches for solving proportions, consider the following tasks:

Tammy bought 3 widgets for $2.40. At the same price, how much would 10 widgets cost?

Tammy bought 4 widgets for $3.75. How much would a dozen widgets cost?

STOP Before reading further, solve these two problems using an approach for each that seems most reasonable to you.

In the first situation, it is perhaps easiest to determine the cost of one widget—the unit rate or unit price. This can be found by dividing the price of three widgets by 3. Multiplying this unit rate of $0.80 per widget by 10 will produce the answer. This approach is referred to as a *unit-rate* method of solving proportions. Notice that the unit rate is a within ratio.

In the second problem, a unit-rate approach could be used, but the division does not appear to be easy. Since 12 is a multiple of 4, it is easier to notice that the cost of a dozen is 3 times the cost of 4. This is called a *factor-of-change* method. It could have been used on the first problem but would have been awkward. The factor of change between 3 and 10 is $3\frac{1}{3}$. Multiplying $2.40 by $3\frac{1}{3}$ will produce the correct answer. Although the factor-of-change method is a useful way to think about proportions, it is most frequently used when the numbers are compatible. Students should be given problems in which the numbers lend themselves to both approaches so that they will explore both methods. The factor of change is a between ratio.

For each of the following two problems, place the numbers in a little drawing of two ratios in Figure 18.15. Solve each problem. Think about within or between ratios matching up. Do not use cross-multiplication.

At the Office Super Store, you can buy plain #2 pencils, four for 59 cents. The store also sells the same pencils in a large box of 5 dozen pencils for $7.79. How much do you save by buying the large box?

The price of a box of 2 dozen candy bars is $4.80. Bridget wants to buy 5 candy bars. What will she have to pay?

To solve the pencil problem, you might notice that the between ratio of pencils to pencils is 4 to 60, or 1 to 15. If you multiply the 59 cents by 15, the factor of change, you will get the price of the box of 60 if the pencils were sold at the same price. In the candy problem, the within ratio of 24 to $4.80 is easy to use to get the unit rate of 20 cents per candy bar. But what do you do if the numbers don't "come out nicely" like they do in these problems?

Try solving the same problems with new numbers that do not work out so easily. If you are having difficulty with the new problems, discuss them with a friend.

STOP Try the following problem. Make a little sketch as before, and use a technique you have figured out yourself. (Do this now before reading on.)

Brian can run 5 km in 18.4 minutes. If he keeps on running at the same speed, how far can he run in 23 minutes?

The first situation for your sketch is Brian's 5-km run (5 km and 18.4 minutes). The second situation is the unknown distance and 23 minutes. There are at least two things you might consider, and one is no easier than the other. You could look at the between ratios of minutes to minutes in order to find a factor of change. That is, what do you multiply 18.4 by to get 23? On the calculator, compute 23 ÷ 18.4 to get 1.25, the factor of change. Now 5 km × 1.25 is 6.25 km.

The second possibility is to get a unit rate for the 5 km and multiply by 23. That would mean divide both the 5 and the 18.4 by 18.4 (like simplifying a fraction to a denominator of 1). The calculator yields 0.2717391, or about 0.27 km per minute. Multiply this unit rate by 23 minutes and you get 6.2499993 km. In both cases, the longer distance is 6.25 km.

What is important here is to see how to use multiplication to solve the proportional situation. Further, notice that the calculations are based on ideas already developed. The sketch of the two ratios helps keep things straight and avoids any ambiguous cross-multiplying.

The Cross-Product Algorithm

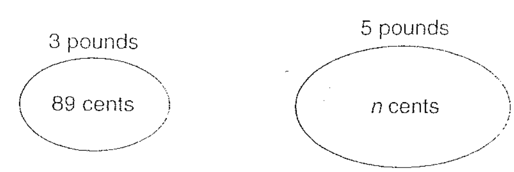
The methods just described come close to being well-defined algorithms, though they are a bit more flexible than cross-product methods. The reality is that the computations involved are exactly the same as in cross-multiplication. Yet some teachers may still want to teach cross-multiplication.

Draw a Simple Model

Given a ratio word problem, the greatest difficulty students have is setting up a correct proportion or equation of two ratios, one of which includes the missing value. "Which fractions do I make? Where does the x go?"

Rather than drill and drill in the hope that they will somehow eventually get it, show students how to sketch a simple picture that will help them determine what parts are related. In Figure 18.16, a simple model is drawn for a typical rate or price problem. The two equations in the figure come from setting up within and between ratios.

Apples are 3 pounds for 89 cents. How much should you pay for 5 pounds?



Within ratios: $\frac{3 \text{ pounds}}{89 \text{ cents}} = \frac{5 \text{ pounds}}{n \text{ cents}}$ or Between ratios: $\frac{3 \text{ pounds}}{5 \text{ pounds}} = \frac{89 \text{ cents}}{n \text{ cents}}$

FIGURE 18.16 A simple drawing helps in a price-to-ratio problem.

Solve the Proportion

Examine the left (within) ratios in the same way as for Brian's 5-km race: Find out what to multiply the left fraction by to get the right. To do this, we would divide 5 by 3 and then multiply that result by 89:

$$\frac{5}{3} \times 89$$

Looking at the same left equation in Figure 18.16, we could also determine the unit price or the price for 1 pound by dividing the 89 cents by 3 and then multiplying this result by 5 to determine the price of 5 pounds:

$$\frac{89}{3} \times 5$$

Now look what happens if we cross-multiply in the original equation:

$$3n = 5 \times 89$$

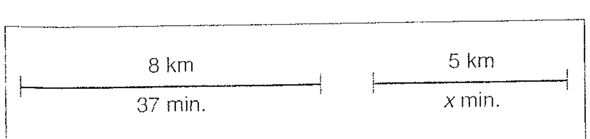
$$n = \frac{5 \times 89}{3}$$

This equation can be solved by dividing the 5 by 3 and multiplying by 89 or dividing 89 by 3 and multiplying by 5. These are exactly the two devices we employed in our more intuitive approach. If you cross-multiply the between ratios, you get exactly the same result. Furthermore, you get the same result if you had written the two ratios inverted, that is, with the reciprocals of each fraction. Try it!

So if you want to develop a cross-product algorithm, it is not unreasonable to do problems like these while encouraging students to use their own methods. If you write out the computations involved, a very small amount of direct teaching can develop the cross-product approach. But why hurry?

In Figure 18.17, a problem involving rates of speed is modeled with simple lines representing the two distances. The distance and the time for each run are modeled with the same line. You cannot see time, but it fits into the distance covered. All equal-rates-of-speed problems can be modeled this way. There really is no significant difference from the drawing used for the apples. Again, it is just as acceptable to write between ratios as within ratios, and students need not worry about which one goes on top as long as the ratios are written in the same order. The model helps with this difficulty.

Jack can run an 8-km race in 37 minutes. If he runs at the same rate, how long should it take him to run a 5-km race?



Within ratios: $\frac{8 \text{ km}}{37 \text{ min.}} = \frac{5 \text{ km}}{x \text{ min.}}$

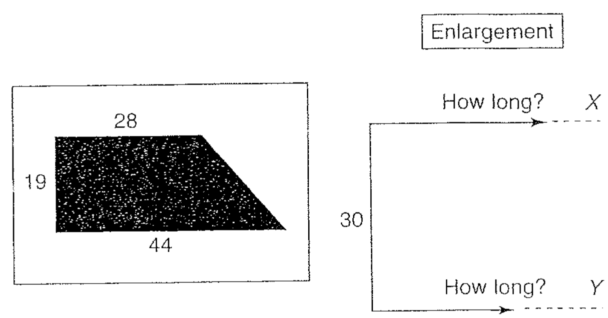
Between ratios: $\frac{8 \text{ km}}{5 \text{ km}} = \frac{37 \text{ min.}}{x \text{ min.}}$

FIGURE 18.17 Line segments can be used to model both time and distance.

Activities Leading to Proportions

In the preceding discussion, simple rate problems were used to help students develop a technique for solving proportions. The next two activities illustrate other common uses of proportional reasoning.

> **ACTIVITY 18.9**
>
> **Scale Drawings**
>
> Provide students with a drawing of a simple geometric figure, including its dimensions. The task is to create a new drawing that is either larger or smaller than the given one. One dimension of the new drawing is specified (see Figure 18.18 for an example). Students can set up between or within ratios and determine the other dimensions by solving the proportion.



Within ratios: $\frac{28}{19} = \frac{X}{30}$ $\quad \frac{19}{44} = \frac{30}{Y}$

Between ratios: $\frac{28}{X} = \frac{19}{30}$ $\quad \frac{19}{30} = \frac{44}{Y}$

FIGURE 18.18 Pictures help in establishing equal ratios.

This scale drawing activity is somewhat simplistic, but it provides students with the essential ideas for setting up proportions. Here are some more interesting situations to consider:

- If you wanted to make a scale model of the solar system and use a Ping-Pong ball for the earth, how far away should the sun be? How large a ball would you need?
- What scale should be used to draw a scale map of your city (or some interesting region) so that it will nicely fit onto a standard piece of poster board?
- Use the scale on a map to estimate the distance and travel time between two points of interest.
- Roll a toy car down a ramp, timing the trip with a stopwatch. How fast was the car traveling in miles per hour? If the speed is proportional to the size of the car, how fast would this have been for a real car?

- Your little sister wants a table and chair for her doll. Her doll is 14 inches tall. How big should you make the table?
- Determine the various distances that a ten-speed bike travels in one turn of the pedals. You will need to count the sprocket teeth on the front and back gears.

Have you ever wondered how scientists estimate the number of bass in a lake or the number of monarch butterflies that migrate each year to Mexico? One method often used is a capture-recapture technique modeled in the next activity.

ACTIVITY 18.10

Capture-Recapture

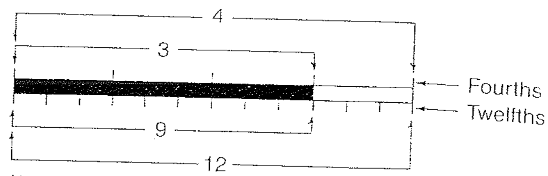
Prepare a shoebox full of some uniform small object such as centicubes or plastic chips. You could also use a larger box filled with Styrofoam packing "peanuts." If the box is your lake and the objects are the fish you want to count, how can you estimate the number without actually counting them? Remember, if they were fish, you couldn't even see them! Have a student reach into the box and "capture" a representative sample of the "fish." For a large box, you may want to capture more than a handful. "Tag" each fish by marking it in some way—marking pen or sticky dot. Count and record the number tagged, and then return them to the box. The assumption of the scientist is that tagged animals will mix uniformly with the larger population, so mix them thoroughly. Next, have five to ten students make a recapture of fish from the box. Each counts the total captured and the number in the capture that are tagged. Accumulate these data.

Now the task is to use all of the information to estimate the number of fish in the lake. The recapture data provide an estimated ratio of tagged to untagged fish. The number tagged to the total population should be in the same ratio. After solving the proportion, have students count the actual items in the box to see how close their estimate is.

For a more detailed description of the "Capture-Recapture" activity, see the NCTM Addenda Series book *Understanding Rational Numbers and Proportions* (Curcio & Bezuk, 1994).

Percent Problems as Proportions

Percent has traditionally been included as a topic with ratio and proportion because percent is one form of ratio, a part-to-whole ratio. In Chapter 17, it was shown that percent problems can be connected to fraction concepts. Here the same part-to-whole fraction concept of percent will be extended to ratio and proportion concepts. Ideally, all of these ideas (fractions, decimals, ratio, proportion, and percent) should be conceptually integrated. The better that students connect these ideas, the more flexible and useful their reasoning and problem-solving skills will be.



Within ratios

$$\frac{\text{Part}}{\text{Whole}} = \frac{3\text{ (fourths)}}{4\text{ (fourths)}} = \frac{9\text{ (twelfths)}}{12\text{ (twelfths)}}$$

FIGURE 18.19 Equivalent fractions as proportions.

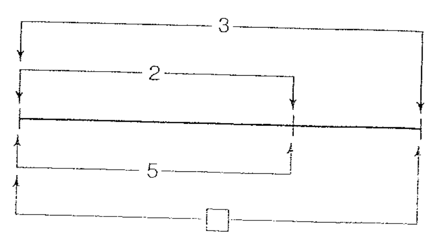
Equivalent Fractions as Proportions

First consider how equivalent fractions can be interpreted as a proportion using the same simple models already used. In Figure 18.19, a line segment is partitioned in two different ways: in fourths on one side and in twelfths on the other. In the previous examples, proportions were established based on two amounts of apples, two different distances or runs, and two different sizes of drawings. Here only one thing is measured—the part of a whole—but it is measured or partitioned two ways: in fourths and in twelfths.

The within ratios are ratios of part to whole within each measurement. Within ratios result in the usual equivalent fraction equation, $\frac{3}{4} = \frac{6}{12}$ (3 fourths are to 4 fourths as 9 twelfths are to 12 twelfths). The between proportion equates a part-to-part ratio with a whole-to-whole ratio, or $\frac{3}{9} = \frac{4}{12}$ (3 fourths are to 9 twelfths as 4 fourths are to 12 twelfths).

A simple line segment drawing similar to the one in Figure 18.19 could be drawn to set up a proportion to solve any equivalent-fraction problem, even ones that do not result in whole-number numerators or denominators. An example is shown in Figure 18.20.

$$\frac{2}{3} = \frac{5}{\square}$$



$$2 \times \square = 3 \times 5$$

$$\square = \frac{15}{2} = 7\frac{1}{2} = 7.5$$

$$\frac{2}{3} = \frac{5}{7\frac{1}{2}} = \frac{5}{7.5}$$

Can you interpret these fractions?

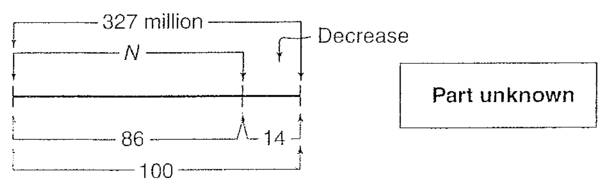
FIGURE 18.20 Solving equivalent-fraction problems as equivalent ratios using cross-products.

Percent Problems

All percent problems are exactly the same as the equivalent-fraction examples. They involve a part and a whole measured in some unit and the same part and whole measured in hundredths—that is, in percents. A simple line segment drawing can be used for each of the three types of percent problems. Using this model as a guide, a proportion can be written and solved by the cross-product algorithm. Examples of each type of problem are shown in Figure 18.21.

It is tempting to teach all percent problems in this one way. Developmentally, such an approach is not recommended. Even though the approach is conceptual, it does not translate easily to intuitive ideas, mental arithmetic, or estimation as discussed in Chapter 17. The modeling and proportion approach of Figure 18.21 is suggested only as a way to help students analyze problems that may verbally present some difficulty. The approach of Chapter 17, which relates percent to part-whole fraction concepts, should probably receive more emphasis.

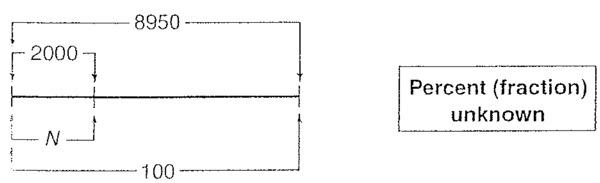
In 1960, U.S. railroads carried 327 million passengers. Over the next 20 years, there was a 14 percent decrease in passengers. How many passengers rode the railroads in 1980?



$$\frac{N}{327 \text{ million}} = \frac{86}{100} = 0.86$$

$N = 0.86 \times 327$ million → about 281 million

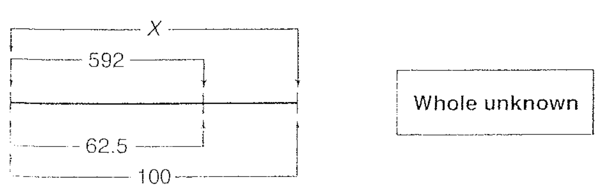
Sylvia's new boat cost $8950. She made a down payment of $2000. What percent of the sales price was Sylvia's down payment?



$$\frac{\$2000}{\$8950} = \frac{N}{100}$$

$8950N = 200{,}000$ → $N = 22.35$, or about 22%

The average dressed weight of a beef steer is 62.5 percent of its weight before being slaughtered. If a dressed steer weighs 592 pounds, how much did it weigh "on the hoof"?



$$\frac{592 \text{ lb.}}{X \text{ lb.}} = \frac{62.5}{100}$$

$62.5X = 59{,}200$ → $X = 947.2$, or about 950 lb.

FIGURE 18.21 The three percentage problems solved by setting up a proportion using a simple line segment model.

Technology Note

You have already seen how spreadsheets can be used to create tables and compute ratios. Dynamic geometry programs allow figures to be scaled to larger or smaller sizes by using either a specified ratio or the dilation feature. This provides an excellent connection between the concepts of similarity and proportion.

Very few computer programs specifically address the issue of proportionality. *CampOS Math* (Pierian Spring Software, 1996) has an activity called the Color Mixer in which the concept of ratio is applied to the mixing of red, green, and blue light to create new colors. By way of example, the student may be asked to set the color mixer so that the blue light is at 30 percent intensity and R:G (the ratio of red to green) = 5:9. There is some potential confusion between types of ratios. The program provides a somewhat realistic use of ratio and some minimal problem solving. ■

Literature Connections

Literature brings an exciting dimension to the exploration of proportional reasoning. Many books and stories discuss comparative sizes; concepts of scale as in maps; giants and miniature people who are proportional to regular people; comparative rates, especially rates of speed; and so on. A book may not appear to explore proportions, and the author may not have had that in mind at all, but comparisons are the stuff of many excellent stories and are at the heart of proportional ideas. The suggestions here are intended to give you an idea of what you might look for.

If You Hopped Like a Frog

Schwartz, 1999

David Schwartz, the author of *How Much Is a Million?* and *If You Made a Million*, has found some wonderful mathematics in nature with this new book. Here Schwartz uses proportional reasoning to determine what it would be like if we had the powers or dimensions of familiar animals. "If you hopped like a frog, you could jump from home plate to first base in one mighty leap." This short picture book contains 12 of these fascinating proportions: if you were as strong as an ant . . . , if you flicked your tongue like a chameleon . . . , and so on. At the end of the book, Schwartz provides some factual data on which the proportions are based. The book could be read at least twice at the start of a lesson, and students could find their own comparisons to calculate. This is a wonderful connection to science. There is no reason the comparisons need be between animals and humans as in this book. The toy-car to real-car ratio is an example of comparisons students could be encouraged to explore using this delightful book as a springboard.

Counting on Frank

Clement, 1991

We will refer to this wonderful book again when we discuss measurement in Chapter 19. It is hard to imagine that more mathematics could spring from 24 pages mostly covered with pictures. But the ideas appeal to all ages, and the potential for good investigations is clearly there for older children. The narrator and his pet dog, Frank, estimate, figure, and ponder interesting facts, usually about large numbers in odd settings (enough green peas to be level with the kitchen tabletop). But three spreads of the book are wonderful fantasies of proportions that could easily inspire an entire unit of proportional reasoning projects.

> "If I had grown at the same speed as the tree—6 feet per year—I'd now be almost 50 feet tall!" How fast do we grow? What if we kept growing at the same rate? How old is the narrator? How old would he be when he is 75 feet tall?
>
> If the mosquito that bothers him were 4 million times bigger . . . What would any common object be like if it were a million times bigger?
>
> If the toaster that shoots toast 3 feet in the air were as big as the house . . .

The Borrowers

Norton, 1953

This is the classic tale of little folk living in the walls of a house. The furnishings and implements are created from odds and ends from the full-size human world. The potential to make scale comparisons is endless.

A similar tale unfolds in Shel Silverstein's poem "One Inch Tall" (1974), in which you are invited to imagine being 1 inch tall.

At the other extreme are stories about giants and dinosaurs, but the concepts of scale are similar. Suppose that a giant were 18 feet tall, or three times the height of a tall man. All linear dimensions (arm span, foot length, etc.) would also be three times that of the man. But the surface area would be 3^2 or 9 times as large, and the giant's volume and hence his weight would be 3^3 or 27 times as much as the man's. That would make the giant weigh about 5400 pounds. With a cross-sectional area of the bones only 9 times more, the bones would not be able to hold the weight. This is one reason there are no real giants. ■

Assessment Notes

As you work with your students in solving proportional reasoning tasks, it will be useful for you to think about the type of reasoning that students are using.

- Do they distinguish between proportional situations and additive or nonproportional ones?
- Are they flexible in the way that they attempt to solve proportions? A nonflexible or algorithmic approach, even if correct, may signal that a student is simply following rules.
- Are there differences in thinking about different types of proportional situations? For example, discrete (countable) items in a proportion are sometimes easier to deal with than continuous quantities such as time, distance, or volume.
- Do students seem to understand rates (miles per hour, inches per yard, dollars per pound) as ratios? How students deal with these ideas reflects the development of their proportional thinking.

Keep in mind that proportional reasoning develops slowly over the middle school years. A first unit of two or three weeks' duration in the sixth grade will not be enough for most children to develop true proportional thought. It will be useful to collect anecdotal evidence related to the questions just listed. If most of your students seem to be at the beginning levels of proportional thought, activities involving selection of equal ratios, constructions, and scaling will be more useful than harder proportion problems. Using numbers that lend themselves to easy relationships will also make it easier for students to describe different methods for solving proportion tasks. ■

Reflections on Chapter 18

Writing to Learn

1. Describe the idea of a ratio in your own words. Explain how your idea fits with each of the following statements
   a. A fraction is a ratio.
   b. Ratios can compare things that are not at all alike.
   c. Ratios can compare two parts of the same whole.
   d. Rates such as prices or speeds are ratios.
2. What is a proportion? For each of the statements in item 1, give an example of a proportion. Also give an example of a comparison that is additive rather than proportional.

3. Much of this chapter is about activities that help students observe ratios and develop proportional reasoning abilities. These activities were grouped into four categories:
   a. Selection of equivalent ratios
   b. Comparisons of ratios
   c. Scaling activities using ratio tables
   d. Construction and measurement

   For each of these four categories, pick one activity presented in the book that you did not do as part of your class experiences. Do the activity and briefly describe how you think the activity would contribute to students' proportional reasoning.
4. What can you say about the graph of a collection of equivalent ratios?
5. Make up a realistic proportional situation that can be solved by a factor-of-change approach and another that can be solved by a unit-rate approach. Explain each.
6. Consider this problem: If 50 gallons of fuel oil cost \$56.95, how much can be purchased for \$100? Draw a sketch to illustrate the proportion, and set up the equation in two different ways. One equation should equate within ratios and the other between ratios.
7. Make up a realistic percentage problem, and set up a proportion. Draw a model to help explain why the proportion makes sense. Illustrate how this method could be used for any of the three types of percentage problems.

For Discussion and Exploration

1. Examine a teacher's edition of a basal textbook for the sixth, seventh, or eighth grade. How is the topic of ratio developed? What is the emphasis? Select one lesson, and write a lesson plan that extends the ideas found on the student pages and actively involves the students.
2. In Chapter 17, the three percent problems were developed around the theme of which element was missing—the part, the whole, or the fraction that related the two. In this chapter, percent is related to proportions, an equality of two ratios with one of these ratios a comparison to 100. How are these two approaches alike? Do you prefer one or the other?

Recommendations for Further Reading

Highly Recommended

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers.* Mahwah, NJ: Lawrence Erlbaum.

Lamon is one of the most prolific researchers and writers on the subject of fractions, ratios, and proportional reasoning. Her work is full of specific practical examples of activities and is freely illustrated with children's work. At the same time this is a serious, research-based, and thought-provoking book. Many of the ideas found in this chapter are adapted from this book and other works by Lamon. Anyone seriously interested in the development of proportional reasoning needs to have this book. There is a companion volume with additional examples to elaborate the ideas found here. (See the other suggested readings that follow.)

Langrall, C. W., & Swafford, J. (2000). Three balloons for two dollars. *Mathematics Teaching in the Middle School, 6,* 254–261.

If you cannot find the Lamon book just mentioned and would like to see an overview of her ideas in a short article, then try this one. The authors describe and give examples of four levels of proportional reasoning using examples from the classroom. A good article on a difficult topic.

Miller, J. L., & Fey, J. T. (2000). Proportional reasoning. *Mathematics Teaching in the Middle School, 5,* 310–313.

This short article describes three proportionality tasks with a total of seven questions that were posed to middle school students. The results compared students in a reform curriculum with those in a traditional curriculum. The interesting tasks are quite reproducible. The discussion is also valuable.

Other Suggestions

Alcaro, P., Alston, A., & Katims, N. (2000). Fractions attack! Children thinking and talking mathematically. *Teaching Children Mathematics, 6,* 562–565.

Ben-Chaim, D., Fey, J. T., Fitzgerald, W. M., Benedetto, C., & Miller, J. (1998). Proportional reasoning among 7th grade students with different curricular experiences. *Educational Studies in Mathematics, 36,* 247–273.

Cai, J., & Sun, W. (2002). Developing students' proportional reasoning: A Chinese perspective. In B. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions* (pp. 195–205). Reston, VA: National Council of Teachers of Mathematics.

Cramer, K., Post, T. R., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 159–178). Old Tappan, NJ: Macmillan.

Kenney, P. A., Lindquist, M. M., & Heffernan, C. L. (2002). Butterflies and caterpillars: Multiplicative and proportional reasoning in the early grades. In B. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions* (pp. 87–99). Reston, VA: National Council of Teachers of Mathematics.

Lamon, S. J. (1999). *More: In-depth discussion of the reasoning activities in "Teaching fractions and ratios for understanding."* Mahwah, NJ: Lawrence Erlbaum.

Lamon, S. J. (2002). Part-whole comparisons with unitizing. In Litwiller, B. (Ed.), *Making sense of fractions, ratios, and proportions* (pp. 79–86). Reston, VA: National Council of Teachers of Mathematics.

Lappan, G., Fitzgerald, W., Winter, M. J., & Phillips, E. (1986). *Middle grades mathematics project: Similarity and equivalent fractions.* Menlo Park, CA: AWL Supplemental.

Lo, J., & Watanabe, T. (1997). Developing ratio and proportion schemes: A story of a fifth grader. *Journal for Research in Mathematics Education, 28,* 216–236.

Middleton, J. A., & Van den Heuvel-Panhuizen, M. (1995). The ratio table. *Mathematics Teaching in the Middle School, 1,* 283–287.

Moss, J. (2002). Percents and proportion at the center: Altering the teaching sequence for rational number. In B. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions* (pp. 109–120). Reston, VA: National Council of Teachers of Mathematics.

Smith, J. P. III. (2002). The development of students' knowledge of fractions and ratios. In B. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions* (pp. 3–17). Reston, VA: National Council of Teachers of Mathematics.

Thompson, D. R., Austin, K. A., & Beckmann, C. E. (2002). Using literature as a vehicle to explore proportional reasoning. In B. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions* (pp. 130–137). Reston, VA: National Council of Teachers of Mathematics.