Chapter 1
Problem Solving

1.1 THE ROLE OF PROBLEM SOLVING IN MATHEMATICS TEACHING AND LEARNING

Solving problems is a practical art, like swimming, or skiing, or playing the piano: you can learn it only by imitation and practice... if you wish to learn swimming you have to go in the water, and if you wish to become a problem solver you have to solve problems.

— George Polya

Mathematics is about solving problems. Many students often believe they can't begin to solve problems because they're completely lost and don't know where to start. They tend to prefer to be shown a solution so that they could memorize it and apply the method to a similar problem hopefully occurring on a test. This might be a short-term remedy that helps for a *particular* problem, but doesn't contribute to gaining generalized problem solving techniques for a variety of situations.

Most states in the United States have now adopted the Common Core Standards for mathematics. The following eight Standards for Mathematical Practice not only guide the grade-level standards, but also promote a spectrum of good mathematical learning habits essential for problem solving proficiency.

The 8 Standards for Mathematical Practice

1. **Make sense of problems and persevere in solving them.**
2. **Reason abstractly and quantitatively.**
3. **Construct viable arguments and critique the reasoning of others.**
4. **Model with mathematics.**
5. **Use appropriate tools strategically.**
6. **Attend to precision.**
7. **Look for and make use of structure.**
8. **Look for and express regularity in repeated reasoning.**

This chapter focuses on content and investigations which encourage learners to find resourceful patterns in problem solving, as opposed to experiencing the general feelings of helplessness often associated with doing mathematics. Most experts don't immediately know how to solve math problems they encounter, but they have learned how to approach 'all' problems in similar ways. To begin learning how to do this; however, one must jump in 'head-first' into problem solving! If not, it's like trying to learn to ride a bicycle or learn to swim by being shown videos of other people doing so, but never practicing the actual activities yourself. Swimming looks so easy, but we know it isn't the first time and requires trial and error, instruction, understanding and practice. There is a large gap between 'watching' and 'doing', and learning mathematics is not very different. The next section begins the journey for prospective teachers to practice *systematic* approaches to problem solving, becoming examples which will one day help manifest the standards for mathematical practice in their own students.

1.2 POLYA'S FOUR STEPS OF PROBLEM SOLVING

Even though the tree of mathematics contains many different branches of vast topics, it so happens that there are common strategies one can learn for how to climb along those branches. But remember, in order to learn these strategies, you need to actually do the climbing yourself. Furthermore, for you as a prospective teacher, it is not enough to be able to solve problems yourself. Your task will include making sense of your students' attempts of solving problems and suggesting a variety of possible approaches to your diversely thinking students. This is why this book emphasizes the importance of multiple solution strategies encountered in problem solving, rather than giving numerous examples of procedures to be memorized. In his 1945 book *How To Solve It*, Polya identified the following four basic principles to guide problem solving:

Polya's Four Steps of Problem Solving

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Look back

Before elaborating on the above steps, it is helpful to review Polya's 'Ten Commandments for Teachers' to better understand how to approach the teaching of problem solving:

1. Be interested in your subject.
2. Know your subject.
3. Try to read the faces of your students; try to see their expectations and difficulties; put yourself in their place.
4. Realize that the best way to learn anything is to discover it by yourself.
5. Give your students not only information, but also know-how, mental attitudes, the habit of methodical work.
6. Let them learn guessing.
7. Let them learn proving.
8. Look out for such features of the problem at hand as may be useful in solving the problems to come - try to disclose the general pattern that lies behind the present concrete situation.
9. Do not give away your whole secret at once - let the students guess before you tell it - let them find out by themselves as much as is feasible.
10. Suggest; do not force information down their throats.

1.2.1 Polya's Four Steps Explained

Step 1 - Understand the Problem

This first step can seem frustrating sometimes when first encountered by students, since if we 'understood' the problem, then everything would be easy, right? Well, Polya was a mathematician, and mathematicians are very precise in how they say things. The first step is to *understand the problem*, not understand how to solve the problem. So, it is not the case that if you understand the problem, you will necessarily know its solution. For example, if you want to paint a room and need to figure out how much paint to buy, there is a lot of understanding needed before one can get to the point of actually doing the computations that will find the answer you are looking for. How many coats of paint is one going to apply? How many square feet does one gallon of the paint cover in one coat? How many square feet of surface does one want to paint? What is the cost of the paint for each gallon? Understanding the problem begins with questions such as these. So, when beginning your understanding of a problem, try and begin forming questions such as:

- Do you know what the problem is asking?
- Do you know all of the terminology and symbols given in the problem statement?

- Do you know how the given information relates to what the problem is asking?
- Can you discuss and restate the problem to someone else?
- Is the problem like another one you might have already encountered?

Step 2 - Devise a Plan

After gaining a sense of what a problem is asking, the next step is to begin organizing problem information so that we can apply our thought and reflection. We do not want to be like the artist with a blank canvas, or the writer with the blank page, suffering from writer's block. We need to look at images, relationships, numbers, or any other relevant objects related to the problem so that we can begin to form a plan or strategy for how to solve the problem.

Some Strategies of Problem Solving When Devising a Plan

1. Guess and check.
2. Make lists.
3. Draw a picture or graph.
4. Look for a pattern.
5. Work backwards.
6. Make a table.
7. Assign variables to quantities.
8. Write an equation.
9. Find a formula.
10. Solve a simpler problem.

Step 3 - Carry out the Plan

At this stage, you have understood what the problem is asking and have assembled the information into some kind of mathematical form, which might be in terms of numbers, equations, graphs or pictures. After reflection and thought about the information given in the problem, a plan usually arises and begins forming. Once a plan is made, it is ready to be tested, or carried out. There is a saying in the game of Chess that "any plan is better than no plan at all." The same goes with solving mathematics problems. It is best not to judge your plan too harshly at first. If it does not work out, you may learn something from that, which can lead you to update or reject the plan for something better. One of the most common mistakes seen by teachers when their students attempt problem solving is that students often jump to the 'Carry Out the Plan' step before they have understood the problem and devised a plan. Doing this makes problem solving a lot harder since it will probably be much more difficult to evaluate the results of the plan if it was not based on a strategy developed from comprehension of what the problem was asking.

Step 4 - Look Back

After a plan has been put forward and implemented, it is time to reflect on the solution to determine if it has solved the problem. 'Looking Back' is easiest when the problem solver has thoroughly understood the problem, since the problem solver then has a good idea if the solution fits what the problem is asking. If the solution does not make sense or seem to solve the problem, it could mean that all previous problem solving steps need work. This points to a 'cycle' of problem solving in which the solver goes back to update their former understanding and strategies used to develop a plan, so that a new plan can be formed and checked against the updated understanding of the problem. Working in this way should ideally lead to a plan which, when checked, solves the problem. 'Looking Back' also includes looking for alternative, possibly simpler, strategies. In addition, you may 'play' with the given information and try to analyze under what conditions the problem would be solvable. This may lead to generalizations so that you will learn to solve a whole family of problems from the one problem you are working on.

1.2.2 Polya's Four Steps in Action

In order to demonstrate this four step problem solving process in action, let's consider the following problem:

There are ducks and rabbits in a yard. Together, they have 12 heads and 30 legs. How many of the animals are ducks and how many are rabbits?

Step 1 - Understand the Problem: It helps to visualize the problem situation. Of course, you have seen ducks and rabbits and know that they all have one head each, but ducks have two legs while rabbits have four. Notice that if all 12 were ducks, they would have only 24 legs, and if all 12 were rabbits, they would have 48 legs. So we know that there must be a mixture of the two kinds of animals in the yard.

Step 2 - Devise a Plan: There are different possible approaches for this problem. Here are a couple.

Plan A.

Let's organize all possible choices for the number of ducks and the consequences of those choices into a table.

number of ducks	number of rabbits	total number of legs
0	12	48
1	11	46
2	10	44
⋮	⋮	⋮

Plan B.

Introduce a variable, say x, for the number of ducks. Express the number of rabbits in terms of x. Write the number of legs all the ducks and rabbits have, and then add them up. Since the total number of legs is given, we will obtain an equation. Then we will solve this equation for x, the number of ducks, and finally, figure out the number of rabbits.

Step 3 - Carry out the Plan:

Plan A.

number of ducks	number of rabbits	total number of legs
0	12	48
1	11	46
2	10	44
3	9	42
4	8	40
5	7	38
6	6	36
7	5	34
8	4	32
9	3	30

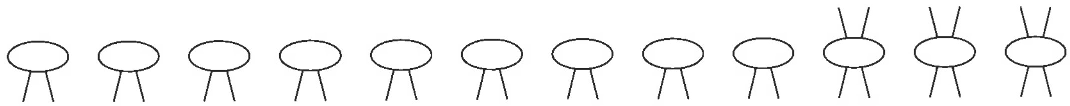
Plan B.

Let x be the number of ducks. Then the number of rabbits is $12-x$. Since each duck has 2 legs, the total number of duck legs is $2x$. Since each rabbit has 4 legs, the total number of rabbit legs is $4(12-x)$. Therefore the total number of all legs is $2x+4(12-x)$. So we have:

$$\begin{aligned} 2x+4(12-x) &= 30 \\ 2x+48-4x &= 30 \\ 18 &= 2x \\ 9 &= x \end{aligned}$$

Thus there are 9 ducks. It follows then that there are 3 rabbits.

Step 4 - Look Back:
Let's check our answer with a drawing.



This suggests another possible solution to the problem: draw 12 'heads' and 2 'legs' for each head. We will have only 24 legs then. But we need 30. So we must draw 6 more legs, thus 'transforming' 3 ducks into rabbits, leaving only 9 ducks.

If you wanted to write a similar problem, could you pick, say, 10 for the number of heads and 22 for the number of legs? How about 16 heads and 70 legs? How about 14 heads and 38 legs? What kind of numbers would work in general? What if you wanted to use different animals, say, ducks and octopuses? What can you say about the number of heads and the number of legs? Would 12 heads and 40 legs work?

Note that we could have chosen the variable x to represent the number of rabbits, rather than ducks. Then the number of ducks would be $12-x$, and the total number of legs would be $4x+2(12-x)$. So we would obtain the following equation:

$$\begin{aligned} 4x+2(12-x) &= 30 \\ 4x+24-2x &= 30 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

So there are 3 rabbits and therefore 9 ducks. Notice that this equation, and its solution ($x=3$) is different from the one we had in Plan B above. This is because our variable x denotes a different quantity here; but, the answer to the problem is the same: 9 ducks, 3 rabbits.

1.2.3 The Benefits of Alternative Solutions

We all have strengths and weaknesses. Solving a problem in different ways can help us learn in our weak areas. For example, if you feel more comfortable working with tables than equations, solving the problem above in both ways may help you understand equations better. The answer should not depend on the method you used to solve a problem. If you solved a problem in two different ways and got two different answers, then you know that at least one solution is wrong.

Since our students often think differently, it helps if we can convey ideas in a variety of ways. It is also crucial for a teacher to learn how to understand various explanations that students come up with.

For example, 29×32 can be calculated in a variety of ways.

1. You may use the multiplication algorithm:

$$\begin{array}{r} 29 \\ \times\ 32 \\ \hline 58 \\ 87 \\ \hline 928 \end{array}$$

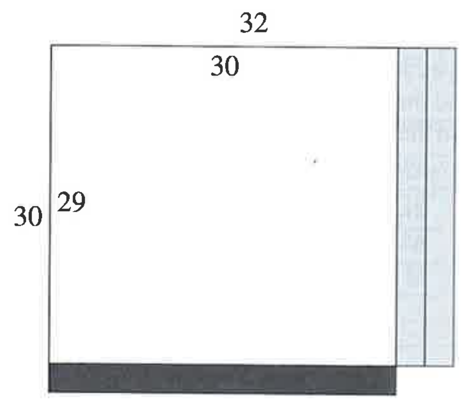
2. You could think of 29 as $30-1$, so 29×32 is 32 less than 30×32:

$$29\times 32=(30-1)\times 32=30\times 32-32=960-32=928$$

3. You could think of 32 as $2\times 2\times 2\times 2\times 2$, so

$$29\times 32 = \underbrace{\underbrace{\underbrace{\underbrace{\underbrace{29\times 2}_{58}\times 2}_{116}\times 2}_{232}\times 2}_{464}\times 2}_{928}$$

4. You may visualize 29×32 as the area of a 29 unit by 32 unit rectangle, and relate it to the area of a 30 unit by 30 unit rectangle.



We need to add two of 1×29 rectangles, then take away one 1×30 rectangle. So the area of the 29×32 rectangle will be $2\times 29-30=28$ more than the area of the 30×30 rectangle:

$$30\times 30+28=928$$