MATH 110 **Homework 11**

The goal of the first problem is to try to come up with an example of a topological space (X, τ) and a subset $A \subseteq X$ such that the sequences

$$A, \quad int(A), \quad \overline{int(A)}, \quad int(\overline{int(A)}), \quad \overline{int(\overline{int(A)})}, \dots \tag{1}$$

and

$$\overline{A}, \quad int(\overline{A}), \quad \overline{int(\overline{A})}, \quad \int(\overline{int(\overline{A})}), \dots \tag{2}$$

give as many different sets as possible (see lecture notes). As was mentioned (but not proved yet) in class and in the lecture notes, the maximum number of different sets possible is seven in each sequence, so 14 sets total. If you can find an example that gives 14 sets, that would be great, but is not required for the homework. Some ideas are given below. Feel free to use or modify them, or to try different ones.

Idea 1. Combine the ideas of using a half open/half closed interval, isolated points, missing points from an interval, and rational/irrational numbers. For example, you could try something like the following: $[1,2) \cup \big([3,\infty) \cap \mathbb{Q}\big)$, $[1,2) \cup \big((2,3] \cap \mathbb{Q}\big) \cup \{4\}$, or even $[1,2) \cup (2,3) \cup \{4\} \cup \big([6,\infty) \cap \mathbb{Q}\big)$.

Idea 2. Create a sequence of intervals that converge to a point, and combine it with some of the previous ideas, e.g. $\{-4\} \cup [-3,-2) \cup (-2,1) \cup \left(\bigcup_{i\in\mathbb{N}} \left(\frac{1}{2n}, \frac{1}{2n-1}\right)\right)$

Idea 3. Choose some other topological space instead of $\mathbb{R}$, e.g. one of the finite sets discussed earlier, and see how this works for some subset (make it an "interesting" subset, not something as simple as the empty set or the whole set).

1. (60%) Choose three sets and work out the sequences (1) and (2) for them. At least one of your sets should be from idea 1 above. The other two can be also from idea 1 or different ones. How many different sets in the two sequences do you get for each choice? Were you able to get 14 different sets?
2. (40%) Consider $\mathbb{R}$ with the usual topology, and the following interpretation: $f(P) = [0,3)$, $f(Q) = \{1\} \cup (2,4)$. Find the following:
   (a) $f(\Box P)$
   (b) $f(\Box\neg P)$
   (c) $f(P \wedge \Box Q)$
   (d) $f(\Box P \to P)$ (Hint: recall that $A \to B$ can be written in terms of negation and disjunction.)