Modal operators. Axioms. Modal logics.

Introduction.

The language of the classical logic is simple, straightforward, and easy to work with. Every statement is either true or false. Given any formula, it is straigtforward to make a truth table and determine whether the formula is a tautology, a contradiction, or neither. Given a set of axioms, it is even possible to write an algorithm that, for any tautology, will derive it from the axioms. However, this language is rather poor, not very expressive. It does not allow statements to change their truth values. For example, the sentence "$x + 2 = 5$" is not a statement since it contains a variable. We called this an open sentence, or a propositional function, over a domain for the variable x. Say, for example, the domain is $\mathbb{N}$. When we put a quantifier, we get a statement: $\forall x \in \mathbb{N}\ x + 2 = 5$, or $\exists x \in \mathbb{N}\ x + 2 = 5$. The language with quantifiers is rich and expressive, but undecidable, in the sense that there is no algorithm that could take any expression and determine whether or not it is true, leave alone write a proof. (This is why we have so many unproved conjectures in mathematics.) A modal logic is a compromise between the two. Consider, for example, the sentence "Dr. Nogin has two children." I this a statement in the classical logic? This is gray area. Yes if you specify the time, but no otherwise. In a sense, this is a propositional function with the variable being time. More presisely, the sentence "Dr. Nogin has two children at time moment t" is a propositional function. For some moments (e.g. at the time you are reading these notes) it is true and for some times (in the past) it is false (since Dr. Nogin didn't always have two children).

A modal logic is an extension of the classical logic with more operators. There are many different modal logics depending on the interpretation of these additional operators and their accepted properties (axioms). Modal logics are more expressive than the classical logic, but decidable: we can write a finite list of axioms and it is possible to write an algorithm that would take a formula and determine whether it can be derived from these axioms; in the latter case, the algorithm would show how it is derived.

Modal operators and their interpretations.

The first, basic, modal operator is $\Box$, called box, or square, or necessity.

Given a formula F, the formula $\Box F$, depending on interpretation/application, could mean:

- necessarily F
- F is always true
- F has always been true
- F will always be true
- F is derivable/provable
- F is true everywhere on a subsequent path (in computer science)
- F is true in every possible accessible world (in philosophy)

The modal operator $\Diamond$, called diamond, or possibility, is defined as the dual of $\Box$, i.e. by

$$\Diamond F = \neg\Box\neg F.$$

Given a formula F, the formula $\Diamond F$, depending on interpretation/application, could mean:

- possibly F
- F is sometimes true
- F has sometimes been true
- F will sometimes be true
- $\neg F$ is not derivable/provable
- F is true somewhere on a subsequent path (in computer science)
- F is true in some possible accessible world (in philosophy)

The operators $\Box$ and $\Diamond$ should remind you of the quantifiers $\forall$ and $\exists$, and, in fact, they share a lot of properties with these quantifiers, but they are simplier than the quantifiers.

Axiom K.

The following axiom is considered the basic axiom for all modal logics, in the sense that for all useful interpretations of the operator $\Box$, this property must be satisfied.

K: $\quad \Box(F \to G) \to (\Box F \to \Box G)$

For example, we could think of this axiom as follows. If $\Box F$ means "F is always true," then the axiom says that if F always implies G, then if F is always true, then G is always true. Or, if $\Box F$ means "F is derivable," then the axiom says that if $F \to G$ is derivable, then if F is derivable, then G is derivable (this should remind you of Modus Ponens).

In addition to Modus Ponens $\dfrac{F, F \to G}{G}$, in modal logic we also have the following inference rule.

Necessitation rule: $\quad \dfrac{F}{\Box F}$.

This rule means that if we derived F, then we say that we derived $\Box F$ as well. This makes sense because if F is derivable, then we want to say that "F is derivable" is derivable.

Theorem. From the axioms of the classical logic and axiom K, using Modus Ponens and Necessitation, we can derive the following formulas:

$\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$,

$\Box A \vee \Box B \to \Box(A \vee B)$,

$\Diamond A \vee \Diamond B \leftrightarrow \Diamond A \vee \Diamond B$,

$\Diamond(A \wedge B) \to \Diamond A \wedge \Diamond B$.

Note: the proof of the above is rather long and is omitted in this class.

Notice the similarity between these and the following properties we saw for the quantifiers:

$\forall x\ (P(x) \wedge Q(x)) \leftrightarrow (\forall x\ P(x) \wedge \forall x\ Q(x))$,

$\forall x\ P(x) \vee \forall x\ Q(x) \to \forall x\ (P(x) \vee Q(x))$,

$\exists x\ P(x) \vee \exists x\ Q(x) \leftrightarrow \exists x\ P(x) \vee \exists x\ Q(x)$,

$\exists x\ (P(x) \wedge Q(x)) \to \exists x\ P(x) \wedge \exists x Q(x)$.

More axioms.

The following axioms make sense in various interpretations/applications, and have been studied by mathematicians, computer scientists, and philosophers. The first three, for example, make sense if we interpret $\Box F$ as "F is always true" and $\Diamond F$ as "F is sometimes true."

K: $\Box(F \to G) \to (\Box F \to \Box G)$

(This axiom is already explained above.)

T: $\Box F \to F$

(If F is always true, then F.)

4: $\Box F \to \Box\Box F$

(If F is always true, then the statement "F is always true" is always true.)

D: $\Box F \to \Diamond F$

B: $F \to \Box\Diamond F$

5: $\Diamond F \to \Box\Diamond F$

Modal logics.

All modal logics listed below contain all axioms of the classical logic (equivalently, all tautologies), and Modus Ponens and Necessitation rules. Additional axioms are given in the table.

Logic	Additional axioms
K	K
D	K, D
T	K, T
B	K, T, B
K4	K, 4
D4	K, D, 4
S4	K, T, 4
S5	K, T, 5 or K, D, 4, B

Theorem. Axiom D can be derived in logic T.

Proof.

1. $\Box F \to F$ (axiom T)
2. $\Box \neg F \to \neg F$ (axiom T)
3. $F \to \neg \Box \neg F$ (contrapositive of step 2.)
4. $\Box F \to \neg \Box \neg F$ (steps 1, 3, and transitivity)

Remark: by contrapositive above we mean the tautology $(P \to Q) \to (\neg Q \to \neg P)$, and by transitivity we mean the tautology $((P \to Q) \wedge (Q \to R)) \to (P \to R)$.

We say that logic D is weaker than logic T, meaning that everything that is derivable in D, is also derivable in T.

Also, logic K is weaker than the other ones, logic K4 is weaker than D4, S4, and S5, and so on.

We also say that K is a sublogic of the other logics listed above, D is a sublogic of T, and so on. Below is a diagram that shows the sublogic relationship.