Interpretations of modal logics.

Definition. An interpretation of a modal logic in a topological space (X, τ) is a function

$$f : \{\text{Formulas}\} \to \{\text{Subsets of } X\}$$

that satisfies:

$$\begin{aligned} f(\neg F_1) &= \overline{f(F_1)}, \\ f(F_1 \vee F_2) &= f(F_1) \cup f(F_2), \\ f(F_1 \wedge F_2) &= f(F_1) \cap f(F_2), \\ f(\Box F_1) &= int(f(F_1)). \end{aligned}$$

Example. Let $X = \mathbb{R}$, with the usual topology, and suppose f is an interpretation such that $f(P) = (0, 1)$ and $f(Q) = [1, 2)$. Then

$$\begin{aligned} f((\neg(\Box Q)) \wedge P) &= f(\neg(\Box Q)) \cap f(P) \\ &= \overline{f(\Box Q)} \cap f(P) \\ &= \overline{int(f(Q))} \cap f(P) \\ &= \overline{(1, 2)} \cap (0, 1) \\ &= ((-\infty, 1] \cup [2, \infty)) \cap (0, 1) \\ &= (0, 1). \end{aligned}$$

Remark. $f(\Diamond P) = f(\neg\Box\neg P) = \overline{int(\overline{f(P)})} = cl(f(P))$, the closure of $f(P)$.

Definition. A formula F_1 is called valid in a topological space (X, τ) if for any interpretation f in (X, τ), we have $f(F_1) = X$.

Definition. We say that a formula F_1 is valid over all topological spaces if it is valid in any topological space (X, τ).

Lemma. If X is a set and A and B are subsets of X, then $A \subseteq B$ if and only if $\overline{A} \cup B = X$.

Lemma. An implication $F_1 \to F_2$ (where F_1 and F_2 are some expressions) is valid in a topological space if for any interpretation f, we have $f(F_1) \subseteq f(F_2)$.

Theorem. Axioms K, T, and 4 are valid in all topological spaces.

Proof. Axiom T is valid because for any subset $A \subseteq X$, we have $int(A) \subseteq A$, thus

$$f(\Box P) = int(f(P)) \subseteq f(P).$$

Axiom 4 is valid because for any subset $A \subseteq X$, we have $int(A) = int(int(A))$, thus

$$f(\Box P) = int(f(P)) = int(int(f(P)) = f(\Box\Box P).$$

Axiom K is a bit harder to prove, and the proof is omitted in this class.

Theorem.

1. If both F_1 and $F_1 \to F_2$ are valid in a topological space, then F_2 is valied also.
2. If F_1 is valid in a topological space, then $\Box F_1$ is valid also.

Corollary. Any formula that is derivable in S4 (i.e. is derivable from the axioms of the classical logic, K, T, and 4 using the Modus Ponens and Necessitation rules), is valid over all topological spaces.

Theorem. (much harder to prove, so not proved in this class) Any formula that is valid over all topological spaces is derivable in S4.

Remark. We say that S4 is sound and complete over all topological spaces.

Corollary. Axiom D is valid over all topological spaces, since it is derivable in S4.

Remark. Axioms B and 5 are not derivable in S4, so they are not valid over all topological spaces. Each of them is valid in some topological spaces though.

Example. Consider any nonempty set X with discrete topology (i.e. every subset of X is open, and therefore every subset is also closed). Then $f(\Box P) = int(f(P)) = f(P)$ and $f(\Diamond P) = cl(f(P)) = f(P)$. Therefore, $f(\Box\Diamond P) = f(P)$ also. It follows that both axioms B and 5 are valid, since $f(P) \subseteq f(\Box\Diamond P)$ and $f(\Diamond P) \subseteq f(\Box\Diamond P)$.

Remark. Suppose that some formula of the form $F_1 \to F_2$ is derivable in S4. Then we have:

(1) $\Box(F_1 \to F_2)$ (necessitation Rule)

(2) $\Box(F_1 \to F_2) \to (\Box F_1 \to \Box F_2)$ (axiom K)

(3) $\Box F_1 \to \Box F_2$ (steps (1), (2), and Modus Ponens)

Getting from $F_1 \to F_2$ to $\Box F_1 \to \Box F_2$ is referred to as "applying the box."

Interior/complement problem.

Recall that for our interior/complement problem, we wanted to prove that

$$int\left(\overline{int\left(\overline{int(\overline{int(A)})}\right)}\right) = int\left(\overline{int(A)}\right).$$

Instead, we will prove that in S4,

$$\Box\neg\Box\neg\Box\neg\Box P \equiv \Box\neg\Box P.$$

Make the following substitution: $R = \neg P$ (so, $P = \neg R$), then the above becomes

$$\Box\neg\Box\neg\Box\neg\Box\neg R \equiv \Box\neg\Box\neg R.$$

Since $\neg\Box\neg$ is the same as $\Diamond$, we can write this equality as

$$\Box\Diamond\Box\Diamond R \equiv \Box\Diamond R.$$

To show logical equivalence , we will prove $\Box\Diamond\Box\Diamond R \to \Box\Diamond R$ and $\Box\Diamond R \to \Box\Diamond\Box\Diamond R$.

Proof of $\Box\Diamond R \to \Box\Diamond\Box\Diamond R$.

(1) $\Box P \to P$ (axiom T)

(2) $\Box\neg Q \to \neg Q$ (step (1) and substitution $P = \neg Q$)

(3) $\neg\neg Q \to \neg\Box\neg Q$ (contrapositive of (2))

(4) $Q \to \Diamond Q$ (simplify (3))

(5) $\Box S \to \Diamond\Box S$ (step (4) and substitution $Q = \Box S$)

(6) $\Box\Box S \to \Box\Diamond\Box S$ (apply box to (5))

(7) $\Box S \to \Box\Box S$ (axiom 4)

(8) $\Box S \to \Box\Diamond\Box S$ (steps (6) and (7))

(9) $\Box\Diamond R \to \Box\Diamond\Box\Diamond R$ (step (8) and substitution $S = \Diamond R$)

Proof of $\Box\Diamond\Box\Diamond R \to \Box\Diamond R$ is similar and is left as an exercise.