1

Sets

In this initial chapter, you will be introduced to, or more than likely be reminded of, a fundamental idea that occurs throughout mathematics: sets. Indeed, a set is an object from which every mathematical structure is constructed (as we will often see in the succeeding chapters). Although there is a formal subject called set theory in which the properties of sets follow from a number of axioms, this is neither our interest nor our need. It is our desire to keep the discussion of sets informal without sacrificing clarity. It is almost a certainty that portions of this chapter will be familiar to you. Nevertheless, it is important that we understand what is meant by a set, how mathematicians describe sets, the notation used with sets, and several concepts that involve sets.

You've been experiencing sets all your life. In fact, all of the following are examples of sets: the members of a sports team, the items on a shopping list, the integers. As a small child, you learned to say the alphabet. When you did this, you were actually listing the letters that make up the set we call the alphabet. A **set** is a collection of objects. The objects that make up a set are called its **elements** (or **members**). The elements of a softball team are the players, while the elements of the alphabet are letters.

It is customary to use capital (uppercase) letters (such as A, B, C, S, X, Y) to designate sets and lowercase letters (for example, a, b, c, s, x, y) to represent elements of sets. If a is an element of the set A, then we write $a \in A$; if a does not belong to A, then we write $a \notin A$.

1.1 Describing a Set

There will be many occasions when we (or you) will need to describe a set. The most important requirement when describing a set is that the description makes it clear precisely which elements belong to the set.

If a set consists of a small number of elements, then this set can be described by explicitly listing its elements between braces (curly brackets) where the elements are separated by commas. Thus $S = \{1, 2, 3\}$ is a set, consisting of the numbers 1, 2, and 3. The order in which the elements are listed doesn't matter. Thus the set S just mentioned could be written as $S = \{3, 2, 1\}$ or $S = \{2, 1, 3\}$, for example. They describe the same set. If a set T consists of the first five letters of the alphabet, then it is not essential that we write $T = \{a, b, c, d, e\}$, that is, the elements of T need not be listed in alphabetical order. On the other hand, listing the elements of T in any other order may create unnecessary confusion.

The set A of all people who signed the Declaration of Independence and later became president of the United States is $A = \{$John Adams, Thomas Jefferson$\}$ and the set B of all positive even integers less than 20 is $B = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$. Some sets contain too many elements to be listed this way. Perhaps even the set B just given contains too many elements to describe in this manner. In such cases, the ellipsis or "three dot notation" is often helpful. For example, $X = \{1, 3, 5, \dots, 49\}$ is the set of all positive odd integers less than 50, while $Y = \{2, 4, 6, \dots\}$ is the set of all positive even integers. The three dots mean "and so on" for Y and "and so on up to" for X. A set need not contain any elements. Although it may seem peculiar to consider sets without elements, these kinds of sets occur surprisingly often and in a variety of settings. For example, if S is the set of real number solutions of the equation $x^2 + 1 = 0$, then S contains no elements. There is only one set that contains no elements, and it is called the **empty set** (or sometimes the **null set** or **void set**). The empty set is denoted by $\emptyset$. We also write $\emptyset = \{\ \}$. In addition to the example given above, the set of all real numbers x such that $x^2 < 0$ is also empty.

The elements of a set may in fact be sets themselves. The symbol ♦ below indicates the conclusion of an example.

Example 1.1 *The set $S = \{1, 2, \{1, 2\}, \emptyset\}$ consists of four elements, two of which are sets, namely, $\{1, 2\}$ and $\emptyset$. If we write $C = \{1, 2\}$, then we can also write $S = \{1, 2, C, \emptyset\}$.*

The set $T = \{0, \{1, 2, 3\}, 4, 5\}$ also has four elements, namely, the three integers $0, 4$, and 5, and the set $\{1, 2, 3\}$. Even though $2 \in \{1, 2, 3\}$, the number 2 is not an element of T; that is, $2 \notin T$. ♦

Often sets consist of those elements satisfying some condition or possessing some specified property. In this case, we can define such a set as $S = \{x : p(x)\}$, where, by this, we mean that S consists of all those elements x satisfying some condition $p(x)$ concerning x. Some mathematicians write $S = \{x \mid p(x)\}$; that is, some prefer to write a vertical line rather than a colon (which, by itself here, is understood to mean "such that"). For example, if we are studying real number solutions of equations, then

$$S = \{x : (x - 1)(x + 2)(x + 3) = 0\}$$

is the set of all real numbers x such that $(x - 1)(x + 2)(x + 3) = 0$, that is, S is the solution set of the equation $(x - 1)(x + 2)(x + 3) = 0$. We could have written $S = \{1, -2, -3\}$; however, even though this way of expressing S is apparently simpler, it does not tell us that we are interested in the solutions of an equation. The **absolute value** $|x|$ of a real number x is x if $x \geq 0$; while $|x| = -x$ if $x < 0$. Therefore,

$$T = \{x : |x| = 2\}$$

is the set of all real numbers having absolute value 2; that is, $T = \{2, -2\}$. In the sets S and T that we have just described, we understand that "x" refers to a real number x. If there is a possibility that this wouldn't be clear to the reader, then we should specifically say that x is a real number. We'll say more about this soon. The set

$$P = \{x : x \text{ has been a president of the United States}\}$$

describes, rather obviously, all those individuals who have been president of the United States. So Abraham Lincoln belongs to P, but Benjamin Franklin does not.

Example 1.2 *Let $A = \{3, 4, 5, \ldots, 20\}$. If B denotes the set consisting of those elements of A that are less than 8, then we can write*

$$B = \{x \in A : x < 8\} = \{3, 4, 5, 6, 7\}. \quad \blacklozenge$$

Some sets are encountered so often that they are given special notation. We use $\mathbf{N}$ to denote the set of all **positive integers** (or **natural numbers**); that is, $\mathbf{N} = \{1, 2, 3, \ldots\}$. The set of all integers (positive, negative, and zero) is denoted by $\mathbf{Z}$. So $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. With the aid of the notation we've just introduced, we can now describe the set $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ of even integers by

$$E = \{y : y \text{ is an even integer}\} \text{ or } E = \{2x : x \text{ is an integer}\}, \text{ or as}$$

$$E = \{y : y = 2x \text{ for some } x \in \mathbf{Z}\} \text{ or } E = \{2x : x \in \mathbf{Z}\}.$$

Also,

$$S = \{x^2 : x \text{ is an integer}\} = \{x^2 : x \in \mathbf{Z}\} = \{0, 1, 4, 9, \ldots\}$$

describes the set of squares of integers.

The set of **real numbers** is denoted by $\mathbf{R}$ and the set of positive real numbers is denoted by $\mathbf{R}^+$. A real number that can be expressed in the form $\frac{m}{n}$, where $m, n \in \mathbf{Z}$ and $n \neq 0$, is called a **rational number**. For example, $\frac{2}{3}$, $\frac{-5}{11}$, $17 = \frac{17}{1}$, and $\frac{4}{6}$ are rational numbers. The set of all rational numbers is denoted by $\mathbf{Q}$. Of course, $\frac{4}{6} = \frac{2}{3}$. A real number that is not rational is called **irrational.** The real numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{2}$, π, and e are known to be irrational; that is, none of these numbers can be expressed as the ratio of two integers. It is also known that the real numbers with infinite nonrepeating decimal expansions are precisely the irrational numbers. There is no common symbol to denote the set of irrational numbers. We will use $\mathbf{I}$ for the set of all irrational numbers. Thus, $\sqrt{2} \in \mathbf{R}$ and $\sqrt{2} \notin \mathbf{Q}$; so $\sqrt{2} \in \mathbf{I}$.

For a set S, we write $|S|$ to denote the number of elements in S. The number $|S|$ is also referred to as the **cardinal number** or **cardinality** of S. If $A = \{1, 2\}$ and $B = \{1, 2, \{1, 2\}, \emptyset\}$, then $|A| = 2$ and $|B| = 4$. Also, $|\emptyset| = 0$. Although the notation is identical for the cardinality of a set and the absolute value of a real number, we should have no trouble distinguishing between the two. A set S is **finite** if $|S| = n$ for some nonnegative integer n. A set S is **infinite** if it is not finite. For the present, we will use the notation $|S|$ only for finite sets S. In Chapter 10, we will discuss the cardinality of infinite sets.

Let's now consider a few examples of sets that are defined in terms of the special sets we have just described.

Example 1.3 *Let* $D = \{n \in \mathbf{N} : n \leq 9\}$, $E = \{x \in \mathbf{Q} : x \leq 9\}$, $H = \{x \in \mathbf{R} : x^2 - 2 = 0\}$, *and* $J = \{x \in \mathbf{Q} : x^2 - 2 = 0\}$.

(a) Describe the set D by listing its elements.

(b) Give an example of three elements that belong to E but do not belong to D.

(c) Describe the set H by listing its elements.

(d) Describe the set J in another manner.

(e) Determine the cardinality of each set D, H, and J.

Solution

(a) $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(b) $\frac{7}{5}, 0, -3$.

(c) $H = \{\sqrt{2}, -\sqrt{2}\}$.

(d) $J = \emptyset$.

(e) $|D| = 9$, $|H| = 2$, and $|J| = 0$. ♦

A **complex number** is a number of the form $a + bi$, where $a, b \in \mathbf{R}$ and $i = \sqrt{-1}$. A complex number $a + bi$, where $b = 0$, can be expressed as $a + 0i$ or, more simply, as a. Hence $a + 0i = a$ is a real number. Thus every real number is a complex number. Let $\mathbf{C}$ denote the set of complex numbers. If $K = \{x \in \mathbf{C} : x^2 + 1 = 0\}$, then $K = \{i, -i\}$. Of course, if $L = \{x \in \mathbf{R} : x^2 + 1 = 0\}$, then $L = \emptyset$. You might recall that the sum of two complex numbers $a + bi$ and $c + di$ is $(a + c) + (b + d)i$, while their product is

$$(a + bi) \cdot (c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

The special sets that we've just described are now summarized as follows:

symbol	**for the set of**
N	natural numbers (positive integers)
Z	integers
Q	rational numbers
I	irrational numbers
R	real numbers
C	complex numbers

1.2 Subsets

A set A is called a **subset** of a set B if every element of A also belongs to B. If A is a subset of B, then we write $A \subseteq B$. If A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. This property of subsets might remind you of the property of real numbers where if $a, b, c \in \mathbf{R}$ such that if $a \leq b$ and $b \leq c$, then $a \leq c$. For the sets $X = \{1, 3, 6\}$ and $Y = \{1, 2, 3, 5, 6\}$, we have $X \subseteq Y$. Also, $\mathbf{N} \subseteq \mathbf{Z}$ and $\mathbf{Q} \subseteq \mathbf{R}$. In addition, $\mathbf{R} \subseteq \mathbf{C}$. Since $\mathbf{Q} \subseteq \mathbf{R}$ and $\mathbf{R} \subseteq \mathbf{C}$, it therefore follows that $\mathbf{Q} \subseteq \mathbf{C}$. Moreover, every set is a subset of itself.

Example 1.4 *Find two sets* A *and* B *such that* A *is both an element and a subset of* B.

Solution Suppose that we seek two sets A and B such that $A \in B$ and $A \subseteq B$. Let's start with a simple example for A, say $A = \{1\}$. Since we want $A \in B$, the set B must contain the set $\{1\}$ as one of its elements. On the other hand, we also require that $A \subseteq B$, so every element of A must belong to B. Since 1 is the only element of A, it follows that B must also contain the number 1. A possible choice for B is then $B = \{1, \{1\}\}$, although $B = \{1, 2, \{1\}\}$ would also satisfy the conditions. ♦

If a set C is *not* a subset of a set D, then we write $C \not\subseteq D$. In this case, there must be some element of C that is not an element of D. One consequence of this is that the empty set $\emptyset$ is a subset of every set. If this were not the case, then there must be some set A such that $\emptyset \not\subseteq A$. But this would mean there is some element, say x, in $\emptyset$ that is not in A. However, $\emptyset$ contains no elements. So $\emptyset \subseteq A$ for *every* set A.

Example 1.5 *Let* $S = \{1, \{2\}, \{1, 2\}\}$.

(a) Determine which of the following are elements of S:
$1, \{1\}, 2, \{2\}, \{1, 2\}, \{\{1, 2\}\}$.

(b) Determine which of the following are subsets of S:
$\{1\}, \{2\}, \{1, 2\}, \{\{1\}, 2\}, \{1, \{2\}\}, \{\{1\}, \{2\}\}, \{\{1, 2\}\}$.

Solution

(a) The following are elements of S: $1, \{2\}, \{1, 2\}$.

(b) The following are subsets of S: $\{1\}, \{1, \{2\}\}, \{\{1, 2\}\}$. ♦

In a typical discussion of sets, we are ordinarily concerned with subsets of some specified set U, called the **universal set.** For example, we may be dealing only with integers, in which case the universal set is $\mathbf{Z}$, or we may be dealing only with real numbers, in which case the universal set is $\mathbf{R}$. On the other hand, the universal set being considered may be neither $\mathbf{Z}$ nor $\mathbf{R}$. Indeed, U may not even be a set of numbers.

Some frequently encountered subsets of $\mathbf{R}$ are the so-called "intervals". For $a, b \in \mathbf{R}$ and $a < b$, the **open interval** (a, b) is the set

$$(a, b) = \{x \in \mathbf{R} : a < x < b\}.$$

Therefore, all of the real numbers $\frac{5}{2}, \sqrt{5}, e, 3, \pi, 4.99$ belong to $(2, 5)$, but none of the real numbers $\sqrt{2}, 1.99, 2, 5$ belong to $(2, 5)$.

For $a, b \in \mathbf{R}$ and $a \leq b$, the **closed interval** $[a, b]$ is the set

$$[a, b] = \{x \in \mathbf{R} : a \leq x \leq b\}.$$

While $2, 5 \notin (2, 5)$, we do have $2, 5 \in [2, 5]$. The "interval" $[a, a]$ is therefore $\{a\}$. Thus, for $a < b$, we have $(a, b) \subseteq [a, b]$. For $a, b \in \mathbf{R}$ and $a < b$, the **half-open** or **half-closed intervals** $[a, b)$ and $(a, b]$ are defined as expected:

$$[a, b) = \{x \in \mathbf{R} : a \leq x < b\} \text{ and } (a, b] = \{x \in \mathbf{R} : a < x \leq b\}.$$

For $a \in \mathbf{R}$, the infinite intervals $(-\infty, a)$, $(-\infty, a]$, (a, ∞), and $[a, \infty)$ are defined as

$$(-\infty, a) = \{x \in \mathbf{R} : x < a\}, \quad (-\infty, a] = \{x \in \mathbf{R} : x \leq a\},$$
$$(a, \infty) = \{x \in \mathbf{R} : x > a\}, \quad [a, \infty) = \{x \in \mathbf{R} : x \geq a\}.$$

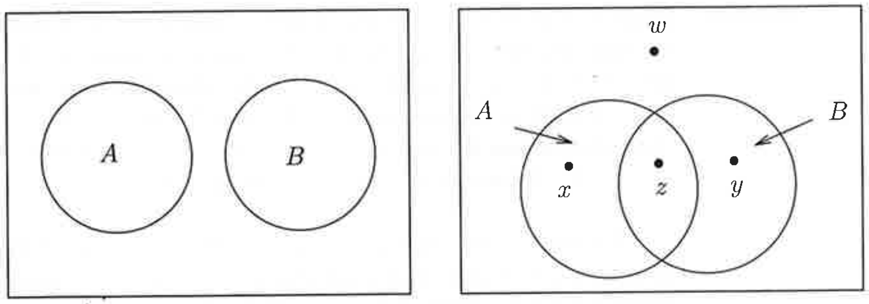

Figure 1.1 Venn diagrams for two sets A and B

The interval $(-\infty, \infty)$ is the set $\mathbf{R}$. Note that the infinity symbols ∞ and $-\infty$ are not real numbers; they are used only to help describe certain intervals. Therefore, $[1, \infty]$, for example, has no meaning.

Two sets A and B are **equal**, indicated by writing $A = B$, if they have exactly the same elements. Another way of saying $A = B$ is that every element of A is in B and every element of B is in A, that is, $A \subseteq B$ and $B \subseteq A$. This fact will be very useful to us in Chapter 4. If $A \neq B$, then there must be some element that belongs to one of A and B but does not belong to the other.

It is often convenient to represent sets by diagrams called **Venn diagrams.** For example, Figure 1.1 shows Venn diagrams for two sets A and B. The diagram on the left represents two sets A and B that have no elements in common, while the diagram on the right is more general. The element x belongs to A but not to B; the element y belongs to B but not to A; the element z belongs to both A and B; and w belongs to neither A nor B. In general, the elements of a set are understood to be those displayed within the region that describes the set. A rectangle in a Venn diagram represents the universal set in this case. Since every element under consideration belongs to the universal set, each element in a Venn diagram lies within the rectangle.

A set A is a **proper subset** of a set B if $A \subseteq B$ but $A \neq B$. If A is a proper subset of B, then we write $A \subset B$. For example, if $S = \{4, 5, 7\}$ and $T = \{3, 4, 5, 6, 7\}$, then $S \subset T$. (Although we write $A \subset B$ to indicate that A is a proper subset of B, it should be mentioned that some prefer to write $A \subsetneqq B$ to indicate that A is a proper subset of B. Indeed, there are some who write $A \subset B$, rather than $A \subseteq B$, to indicate that A is a subset of B. We will follow the notation introduced above, however.)

The set consisting of all subsets of a given set A is called the **power set** of A and is denoted by $\mathcal{P}(A)$.

Example 1.6 *For each set A below, determine $\mathcal{P}(A)$. In each case, determine $|A|$ and $|\mathcal{P}(A)|$.*

(a) $A = \emptyset$, (b) $A = \{a, b\}$, (c) $A = \{1, 2, 3\}$.

Solution

(a) $\mathcal{P}(A) = \{\emptyset\}$. In this case, $|A| = 0$ and $|\mathcal{P}(A)| = 1$.

(b) $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. In this case, $|A| = 2$ and $|\mathcal{P}(A)| = 4$.

(c) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
In this case, $|A| = 3$ and $|\mathcal{P}(A)| = 8$. ◆

Notice that for each set A in Example 1.6, we have $|\mathcal{P}(A)| = 2^{|A|}$. In fact, if A is any finite set, with n elements say, then $\mathcal{P}(A)$ has 2^n elements; that is,

$$|\mathcal{P}(A)| = 2^{|A|}$$

for every finite set A. (Later we will explain why this is true.)

Example 1.7 *If* $C = \{\emptyset, \{\emptyset\}\}$, *then*

$$\mathcal{P}(C) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}.$$

It is important to note that no two of the sets $\emptyset$, $\{\emptyset\}$, *and* $\{\{\emptyset\}\}$ *are equal. (An empty box and a box containing an empty box are not the same.) For the set C above, it is therefore correct to write*

$$\emptyset \subseteq C, \emptyset \subset C, \emptyset \in C, \{\emptyset\} \subseteq C, \{\emptyset\} \subset C, \{\emptyset\} \in C,$$

as well as

$$\{\{\emptyset\}\} \subseteq C, \{\{\emptyset\}\} \notin C, \{\{\emptyset\}\} \in \mathcal{P}(C).$$ ♦

1.3 Set Operations

Just as there are several ways of combining two integers to produce another integer (addition, subtraction, multiplication, and sometimes division), there are several ways to combine two sets to produce another set. The **union** of two sets A and B, denoted by $A \cup B$, is the set of all elements belonging to A or B; that is,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The use of the word "or" here, and in mathematics in general, allows an element of $A \cup B$ to belong to both A and B. That is, x is in $A \cup B$ if x is in A or x is in B or x is in both A and B. A Venn diagram for $A \cup B$ is shown in Figure 1.2.

Example 1.8 *For the sets* $A_1 = \{2, 5, 7, 8\}$, $A_2 = \{1, 3, 5\}$, *and* $A_3 = \{2, 4, 6, 8\}$, *we have*

$$A_1 \cup A_2 = \{1, 2, 3, 5, 7, 8\},$$
$$A_1 \cup A_3 = \{2, 4, 5, 6, 7, 8\},$$
$$A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 8\}.$$

Also, $\mathbf{N} \cup \mathbf{Z} = \mathbf{Z}$ *and* $\mathbf{Q} \cup \mathbf{I} = \mathbf{R}$. ♦

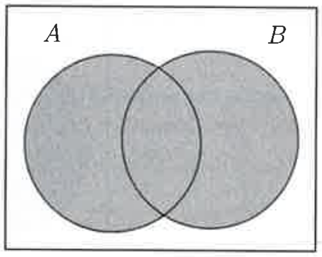

Figure 1.2 A Venn diagram for $A \cup B$

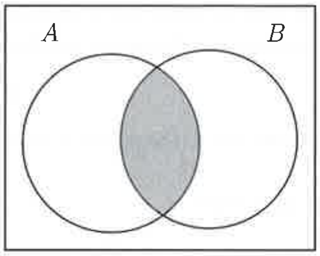

Figure 1.3 A Venn diagram for $A \cap B$

The **intersection** of two sets A and B is the set of all elements belonging to both A and B. The intersection of A and B is denoted by $A \cap B$. In symbols,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

A Venn diagram for $A \cap B$ is shown in Figure 1.3.

Example 1.9 *For the sets A_1, A_2, and A_3 described in Example 1.8,*

$$A_1 \cap A_2 = \{5\},\ A_1 \cap A_3 = \{2, 8\}, \textit{ and } A_2 \cap A_3 = \emptyset.$$

Also, $\mathbf{N} \cap \mathbf{Z} = \mathbf{N}$ *and* $\mathbf{Q} \cap \mathbf{R} = \mathbf{Q}$. ♦

For every two sets A and B, it follows that

$$A \cap B \subseteq A \cup B.$$

If two sets A and B have no elements in common, then $A \cap B = \emptyset$ and A and B are said to be **disjoint**. Consequently, the sets A_2 and A_3 described in Example 1.8 are disjoint; however, A_1 and A_3 are not disjoint since 2 and 8 belong to both sets. Also, $\mathbf{Q}$ and $\mathbf{I}$ are disjoint.

The **difference** $A - B$ of two sets A and B (also written as $A \setminus B$ by some mathematicians) is defined as

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

A Venn diagram for $A - B$ is shown in Figure 1.4.

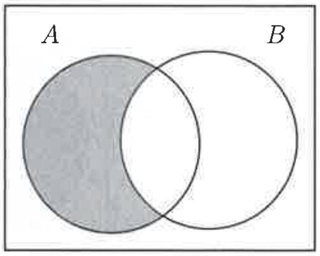

Figure 1.4 A Venn diagram for $A - B$

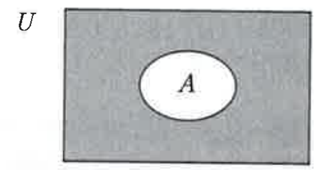

Figure 1.5 A Venn diagram for $\overline{A}$

Example 1.10 *For the sets* $A_1 = \{2, 5, 7, 8\}$ *and* $A_2 = \{1, 3, 5\}$ *in Examples 1.8 and 1.9,* $A_1 - A_2 = \{2, 7, 8\}$ *and* $A_2 - A_1 = \{1, 3\}$*. Furthermore,* $\mathbf{R} - \mathbf{Q} = \mathbf{I}$. ♦

Suppose that we are considering a certain universal set U; that is, all sets being discussed are subsets of U. For a set A, its **complement** is

$$\overline{A} = U - A = \{x : x \in U \text{ and } x \notin A\}.$$

If $U = \mathbf{Z}$, then $\overline{\mathbf{N}} = \{0, -1, -2, \ldots\}$; while if $U = \mathbf{R}$, then $\overline{\mathbf{Q}} = \mathbf{I}$. A Venn diagram for $\overline{A}$ is shown in Figure 1.5.

The set difference $A - B$ is sometimes called the **relative complement** of B in A. Indeed, from the definition, $A - B = \{x : x \in A \text{ and } x \notin B\}$. The set $A - B$ can also be expressed in terms of complements, namely, $A - B = A \cap \overline{B}$. This fact will be established later.

Example 1.11 *Let* $U = \{1, 2, \ldots, 10\}$ *be the universal set,* $A = \{2, 3, 5, 7\}$*, and* $B = \{2, 4, 6, 8, 10\}$*. Determine each of the following:*

(a) $\overline{B}$, (b) $A - B$, (c) $A \cap \overline{B}$, (d) $\overline{\overline{B}}$.

Solution

(a) $\overline{B} = \{1, 3, 5, 7, 9\}$.

(b) $A - B = \{3, 5, 7\}$.

(c) $A \cap \overline{B} = \{3, 5, 7\} = A - B$.

(d) $\overline{\overline{B}} = B = \{2, 4, 6, 8, 10\}$. ♦

Example 1.12 *Let* $A = \{0, \{0\}, \{0, \{0\}\}\}$.

(a) Determine which of the following are elements of A: 0, $\{0\}$, $\{\{0\}\}$.

(b) Determine $|A|$.

(c) Determine which of the following are subsets of A: 0, $\{0\}$, $\{\{0\}\}$.
For (d)–(i), determine the indicated sets.

(d) $\{0\} \cap A$

(e) $\{\{0\}\} \cap A$

(f) $\{\{\{0\}\}\} \cap A$

(g) $\{0\} \cup A$

(h) $\{\{0\}\} \cup A$

(i) $\{\{\{0\}\}\} \cup A$.

Solution

(a) While 0 and $\{0\}$ are elements of A, $\{\{0\}\}$ is not an element of A.

(b) The set A has three elements: 0, $\{0\}$, $\{0, \{0\}\}$. Therefore, $|A| = 3$.

(c) The integer 0 is not a set and so cannot be a subset of A (or a subset of any other set). Since $0 \in A$ and $\{0\} \in A$, it follows that $\{0\} \subseteq A$ and $\{\{0\}\} \subseteq A$.

(d) Since 0 is the only element that belongs to both $\{0\}$ and A, it follows that $\{0\} \cap A = \{0\}$.

(e) Since $\{0\}$ is the only element that belongs to both $\{\{0\}\}$ and A, it follows that $\{\{0\}\} \cap A = \{\{0\}\}$.

(f) Since $\{\{0\}\}$ is not an element of A, it follows that $\{\{\{0\}\}\}$ and A are disjoint sets and so $\{\{\{0\}\}\} \cap A = \emptyset$.

(g) Since $0 \in A$, it follows that $\{0\} \cup A = A$.

(h) Since $\{0\} \in A$, it follows that $\{\{0\}\} \cup A = A$.

(i) Since $\{\{0\}\} \notin A$, it follows that $\{\{\{0\}\}\} \cup A = \{0, \{0\}, \{\{0\}\}, \{0, \{0\}\}\}$. ♦

1.4 Indexed Collections of Sets

We will often encounter situations where more than two sets are combined using the set operations described above. In the case of three sets A, B, and C, the standard Venn diagram is shown in Figure 1.6.

The union $A \cup B \cup C$ is defined as

$$A \cup B \cup C = \{x : x \in A \text{ or } x \in B \text{ or } x \in C\}.$$

Thus, in order for an element to belong to $A \cup B \cup C$, the element must belong to at least one of the sets A, B, and C. Because it is often useful to consider the union of several sets, additional notation is needed. The union of the $n \geq 2$ sets $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \cup A_2 \cup \cdots \cup A_n$ or $\bigcup_{i=1}^{n} A_i$ and is defined as

$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } i,\ 1 \leq i \leq n\}.$$

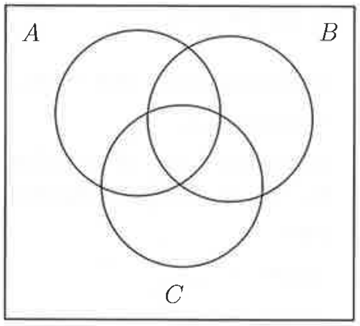

Figure 1.6 A Venn diagram for three sets

Thus, for an element a to belong to $\bigcup_{i=1}^{n} A_i$, it is necessary that a belongs to at least one of the sets $A_1, A_2, \ldots, A_n$.

Example 1.13 *Let* $B_1 = \{1,2\}$, $B_2 = \{2,3\}, \ldots, B_{10} = \{10,11\}$; *that is,* $B_i = \{i, i+1\}$ *for* $i = 1, 2, \ldots, 10$. *Determine each of the following:*

$$\text{(a) } \bigcup_{i=1}^{5} B_i. \quad \text{(b) } \bigcup_{i=1}^{10} B_i. \quad \text{(c) } \bigcup_{i=3}^{7} B_i. \quad \text{(d) } \bigcup_{i=j}^{k} B_i, \text{ where } 1 \le j \le k \le 10.$$

Solution

$$\text{(a) } \bigcup_{i=1}^{5} B_i = \{1, 2, \ldots, 6\}. \quad \text{(b) } \bigcup_{i=1}^{10} B_i = \{1, 2, \ldots, 11\}.$$

$$\text{(c) } \bigcup_{i=3}^{7} B_i = \{3, 4, \ldots, 8\}. \quad \text{(d) } \bigcup_{i=j}^{k} B_i = \{j, j+1, \ldots, k+1\}. \quad \blacklozenge$$

We are often interested in the intersection of several sets as well. The intersection of the $n \ge 2$ sets $A_1, A_2, \ldots, A_n$ is expressed as $A_1 \cap A_2 \cap \cdots \cap A_n$ or $\bigcap_{i=1}^{n} A_i$ and is defined by

$$\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for every } i, 1 \le i \le n\}.$$

The next example concerns the sets mentioned in Example 1.13.

Example 1.14 *Let* $B_i = \{i, i+1\}$ *for* $i = 1, 2, \ldots, 10$. *Determine the following:*

$$\text{(a) } \bigcap_{i=1}^{10} B_i. \quad \text{(b) } B_i \cap B_{i+1}. \quad \text{(c) } \bigcap_{i=j}^{j+1} B_i, \text{ where } 1 \le j < 10.$$

$$\text{(d) } \bigcap_{i=j}^{k} B_i \text{ where } 1 \le j < k \le 10.$$

Solution

$$\text{(a) } \bigcap_{i=1}^{10} B_i = \emptyset. \quad \text{(b) } B_i \cap B_{i+1} = \{i+1\}. \quad \text{(c) } \bigcap_{i=j}^{j+1} B_i = \{j+1\}.$$

$$\text{(d) } \bigcap_{i=j}^{k} B_i = \{j+1\} \text{ if } k = j+1; \text{ while } \bigcap_{i=j}^{k} B_i = \emptyset \text{ if } k > j+1. \quad \blacklozenge$$

There are instances when the union or intersection of a collection of sets cannot be described conveniently (or perhaps at all) in the manner mentioned above. For this reason, we introduce a (nonempty) set I, called an **index set**, which is used as a mechanism for selecting those sets we want to consider. For example, for an index set I, suppose that there is a set S_α for each $\alpha \in I$. We write $\{S_\alpha\}_{\alpha \in I}$ to describe the collection of all sets S_α, where $\alpha \in I$. Such a collection is called an **indexed collection of sets**. We define the union of the sets in $\{S_\alpha\}_{\alpha \in I}$ by

$$\bigcup_{\alpha \in I} S_\alpha = \{x : x \in S_\alpha \text{ for some } \alpha \in I\},$$

and the intersection of these sets by

$$\bigcap_{\alpha \in I} S_\alpha = \{x : x \in S_\alpha \text{ for all } \alpha \in I\}.$$

Hence an element a belongs to $\bigcup_{\alpha \in I} S_\alpha$ if a belongs to at least one of the sets in the collection $\{S_\alpha\}_{\alpha \in I}$, while a belongs to $\bigcap_{\alpha \in I} S_\alpha$ if a belongs to every set in the collection $\{S_\alpha\}_{\alpha \in I}$. We refer to $\bigcup_{\alpha \in I} S_\alpha$ as the union of the collection $\{S_\alpha\}_{\alpha \in I}$ and $\bigcap_{\alpha \in I} S_\alpha$ as the intersection of the collection $\{S_\alpha\}_{\alpha \in I}$. Just as there is nothing special about our choice of i in $\bigcup_{i=1}^{n} A_i$ (that is, we could just as well describe this set by $\bigcup_{j=1}^{n} A_j$, say), there is nothing special about α in $\bigcup_{\alpha \in I} S_\alpha$. We could also describe this set by $\bigcup_{x \in I} S_x$. The variables i and α above are "dummy variables" and any appropriate symbol could be used. Indeed, we could write J or some other symbol for an index set.

Example 1.15 *For $n \in \mathbf{N}$, define $S_n = \{n, 2n\}$. For example, $S_1 = \{1, 2\}$, $S_2 = \{2, 4\}$, and $S_4 = \{4, 8\}$. Then $S_1 \cup S_2 \cup S_4 = \{1, 2, 4, 8\}$. We can also describe this set by means of an index set. If we let $I = \{1, 2, 4\}$, then*

$$\bigcup_{\alpha \in I} S_\alpha = S_1 \cup S_2 \cup S_4. \qquad \blacklozenge$$

Example 1.16 *For each $n \in \mathbf{N}$, define A_n to be the closed interval $[-\frac{1}{n}, \frac{1}{n}]$ of real numbers; that is,*

$$A_n = \left\{x \in \mathbf{R} : -\frac{1}{n} \le x \le \frac{1}{n}\right\}.$$

So $A_1 = [-1, 1]$, $A_2 = [-\frac{1}{2}, \frac{1}{2}]$, $A_3 = [-\frac{1}{3}, \frac{1}{3}]$, and so on. We have now defined the sets $A_1, A_2, A_3, \ldots$. The union of these sets can be written as $A_1 \cup A_2 \cup A_3 \cup \cdots$ or $\bigcup_{i=1}^{\infty} A_i$. Using $\mathbf{N}$ as an index set, we can also write this union as $\bigcup_{n \in \mathbf{N}} A_n$. Since $A_n \subseteq A_1 = [-1, 1]$ for every $n \in \mathbf{N}$, it follows that $\bigcup_{n \in \mathbf{N}} A_n = [-1, 1]$. Certainly, $0 \in A_n$ for every $n \in \mathbf{N}$; in fact, $\bigcap_{n \in \mathbf{N}} A_n = \{0\}$. ♦

Example 1.17 *Let A denote the set of the letters of the alphabet, that is, $A = \{a, b, \ldots, z\}$. For $\alpha \in A$, let A_α consist of α and the two letters that follow α. So $A_a = \{a, b, c\}$ and $A_b = \{b, c, d\}$. By A_y, we will mean the set $\{y, z, a\}$ and $A_z = \{z, a, b\}$. Hence $|A_\alpha| = 3$ for every $\alpha \in A$. Therefore, $\bigcup_{\alpha \in A} A_\alpha = A$. Indeed, if*

$$B = \{a, d, g, j, m, p, s, v, y\},$$

then $\bigcup_{\alpha \in B} A_\alpha = A$ as well. On the other hand, if $I = \{p, q, r\}$, then $\bigcup_{\alpha \in I} A_\alpha = \{p, q, r, s, t\}$; while $\bigcap_{\alpha \in I} A_\alpha = \{r\}$. ♦

Example 1.18 *Let $S = \{1, 2, \ldots, 10\}$. Each of the sets*

$$S_1 = \{1, 2, 3, 4\},\ S_2 = \{4, 5, 6, 7, 8\}, \text{ and } S_3 = \{7, 8, 9, 10\}$$

is a subset of S. Also, $S_1 \cup S_2 \cup S_3 = S$. This union can be described in a number of ways. Define $I = \{1, 2, 3\}$ and $J = \{S_1, S_2, S_3\}$. Then the union of the three sets belonging to

J is precisely $S_1 \cup S_2 \cup S_3$, *which can also be written as*

$$S = S_1 \cup S_2 \cup S_3 = \bigcup_{i=1}^{3} S_i = \bigcup_{\alpha \in I} S_\alpha = \bigcup_{X \in J} X. \quad \blacklozenge$$

1.5 Partitions of Sets

Recall that two sets are disjoint if their intersection is the empty set. A collection $\mathcal{S}$ of subsets of a set A is called **pairwise disjoint** if every two distinct subsets that belong to $\mathcal{S}$ are disjoint. For example, let $A = \{1, 2, \ldots, 7\}$, $B = \{1, 6\}$, $C = \{2, 5\}$, $D = \{4, 7\}$, and $S = \{B, C, D\}$. Then S is a pairwise disjoint collection of subsets of A since $B \cap C = B \cap D = C \cap D = \emptyset$. On the other hand, let $A' = \{1, 2, 3\}$, $B' = \{1, 2\}$, $C' = \{1, 3\}$, $D' = \{2, 3\}$, and $S' = \{B', C', D'\}$. Although S' is a collection of subsets of A' and $B' \cap C' \cap D' = \emptyset$, the set S' is *not* a pairwise disjoint collection of sets since $B' \cap C' \neq \emptyset$, for example. Indeed, $B' \cap D'$ and $C' \cap D'$ are also nonempty.

We will often have the occasion (especially in Chapter 8) to encounter, for a nonempty set A, a collection $\mathcal{S}$ of pairwise disjoint nonempty subsets of A with the added property that every element of A belongs to some subset in $\mathcal{S}$. Such a collection is called a **partition** of A. A **partition** of A can also be defined as a collection $\mathcal{S}$ of nonempty subsets of A such that every element of A belongs to exactly one subset in $\mathcal{S}$. Furthermore, a partition of A can be defined as a collection $\mathcal{S}$ of subsets of A satisfying the three properties:

(1) $X \neq \emptyset$ for every set $X \in \mathcal{S}$;

(2) for every two sets $X, Y \in \mathcal{S}$, either $X = Y$ or $X \cap Y = \emptyset$;

(3) $\bigcup_{X \in \mathcal{S}} X = A$.

Example 1.19 *Consider the following collections of subsets of the set* $A = \{1, 2, 3, 4, 5, 6\}$:

$$S_1 = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\};$$
$$S_2 = \{\{1, 2, 3\}, \{4\}, \emptyset, \{5, 6\}\};$$
$$S_3 = \{\{1, 2\}, \{3, 4, 5\}, \{5, 6\}\};$$
$$S_4 = \{\{1, 4\}, \{3, 5\}, \{2\}\}.$$

Determine which of these sets are partitions of A.

Solution The set S_1 is a partition of A. The set S_2 is not a partition of A since $\emptyset$ is one of the elements of S_2. The set S_3 is not a partition of A either since the element 5 belongs to two distinct subsets in S_3, namely, $\{3, 4, 5\}$ and $\{5, 6\}$. Finally, S_4 is also not a partition of A because the element 6 belongs to no subset in S_4. $\blacklozenge$

As the word "partition" probably suggests, a partition of a nonempty set A is a division of A into nonempty subsets. The partition S_1 of the set A in Example 1.19 is illustrated in the diagram shown in Figure 1.7.

For example, the set $\mathbf{Z}$ of integers can be partitioned into the set of even integers and the set of odd integers. The set $\mathbf{R}$ of real numbers can be partitioned into the set $\mathbf{R}^+$ of positive real numbers, the set of negative real numbers, and the set $\{0\}$ consisting of

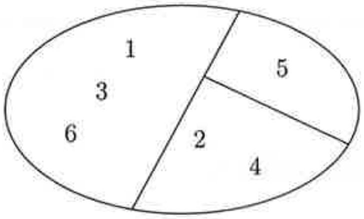

Figure 1.7 A partition of a set

the number 0. In addition, **R** can be partitioned into the set **Q** of rational numbers and the set **I** of irrational numbers.

Example 1.20 *Let* $A = \{1, 2, \ldots, 12\}$.

(a) Give an example of a partition S of A such that $|S| = 5$.

(b) Give an example of a subset T of the partition S in (a) such that $|T| = 3$.

(c) List all those elements B in the partition S in (a) such that $|B| = 2$.

Solution (a) We are seeking a partition S of A consisting of five subsets. One such example is

$$S = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}\}.$$

(b) We are seeking a subset T of S (given in (a)) consisting of three elements. One such example is

$$T = \{\{1, 2\}, \{3, 4\}, \{7, 8, 9\}\}.$$

(c) We have been asked to list all those elements of S (given in (a)) consisting of two elements of A. These elements are: $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$. ♦

1.6 Cartesian Products of Sets

We've already mentioned that when a set A is described by listing its elements, the order in which the elements of A are listed doesn't matter. That is, if the set A consists of two elements x and y, then $A = \{x, y\} = \{y, x\}$. When we speak of the **ordered pair** (x, y), however, this is another story. The ordered pair (x, y) is a single element consisting of a pair of elements in which x is the first element (or first coordinate) of the ordered pair (x, y) and y is the second element (or second coordinate). Moreover, for two ordered pairs (x, y) and (w, z) to be equal, that is, $(x, y) = (w, z)$, we must have $x = w$ and $y = z$. So, if $x \neq y$, then $(x, y) \neq (y, x)$.

The **Cartesian product** (or simply the product) $A \times B$ of two sets A and B is the set consisting of all ordered pairs whose first coordinate belongs to A and whose second coordinate belongs to B. In other words,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Example 1.21 *If* $A = \{x, y\}$ *and* $B = \{1, 2, 3\}$, *then*

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\},$$

while

$$B \times A = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}.$$

Since, for example, $(x, 1) \in A \times B$ *and* $(x, 1) \notin B \times A$, *these two sets do not contain the same elements; so* $A \times B \neq B \times A$. *Also,*

$$A \times A = \{(x, x), (x, y), (y, x), (y, y)\}$$

and

$$B \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}. \quad \blacklozenge$$

We also note that if $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

The Cartesian product $\mathbf{R} \times \mathbf{R}$ is the set of all points in the Euclidean plane. For example, the graph of the straight line $y = 2x + 3$ is the set

$$\{(x, y) \in \mathbf{R} \times \mathbf{R} : \ y = 2x + 3\}.$$

For the sets $A = \{x, y\}$ and $B = \{1, 2, 3\}$ given in Example 1.21, $|A| = 2$ and $|B| = 3$; while $|A \times B| = 6$. Indeed, for all finite sets A and B,

$$|A \times B| = |A| \cdot |B|.$$

Cartesian products will be explored in more detail in Chapter 7.

EXERCISES FOR CHAPTER 1

Section 1.1: Describing a Set

1.1. Which of the following are sets?

(a) $1, 2, 3$
(b) $\{1, 2\}, 3$
(c) $\{\{1\}, 2\}, 3$
(d) $\{1, \{2\}, 3\}$
(e) $\{1, 2, a, b\}$

1.2. Let $S = \{-2, -1, 0, 1, 2, 3\}$. Describe each of the following sets as $\{x \in S : \ p(x)\}$, where $p(x)$ is some condition on x.

(a) $A = \{1, 2, 3\}$
(b) $B = \{0, 1, 2, 3\}$
(c) $C = \{-2, -1\}$.
(d) $D = \{-2, 2, 3\}$.

1.3. Determine the cardinality of each of the following sets:

(a) $A = \{1, 2, 3, 4, 5\}$
(b) $B = \{0, 2, 4, \ldots, 20\}$
(c) $C = \{25, 26, 27, \ldots, 75\}$