Solutions to Odd-Numbered Section Exercises

EXERCISES FOR CHAPTER 1

Section 1.1: Describing a Set

1.1. Only (d) and (e) are sets.

1.3. **(a)** $|A| = 5$, **(b)** $|B| = 11$, **(c)** $|C| = 51$, **(d)** $|D| = 2$, **(e)** $|E| = 1$, **(f)** $|F| = 2$

1.5. **(a)** $A = \{-1, -2, -3, \ldots\} = \{x \in \mathbf{Z} : x \leq -1\}$
(b) $B = \{-3, -2, \ldots, 3\} = \{x \in \mathbf{Z} : -3 \leq x \leq 3\} = \{x \in \mathbf{Z} : |x| \leq 3\}$
(c) $C = \{-2, -1, 1, 2\} = \{x \in \mathbf{Z} : -2 \leq x \leq 2, x \neq 0\} = \{x \in \mathbf{Z} : 0 < |x| \leq 2\}$

1.7. **(a)** $A = \{\cdots, -4, -1, 2, 5, 8, \cdots\} = \{3x + 2 : x \in \mathbf{Z}\}$
(b) $B = \{\cdots, -10, -5, 0, 5, 10, \cdots\} = \{5x : x \in \mathbf{Z}\}$
(c) $C = \{1, 8, 27, 64, 125, \cdots\} = \{x^3 : x \in \mathbf{N}\}$

Section 1.2: Subsets

1.9. Let $r = \min(c - a, b - c)$ and let $I = (c - r, c + r)$. Then I is centered at c and $I \subseteq (a, b)$.

1.11. See Figure 1.

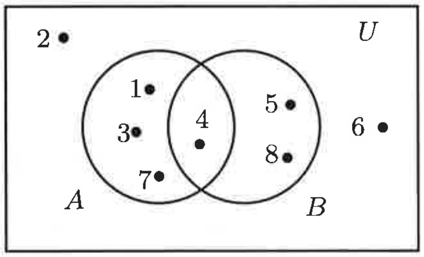

Figure 1 Answer for Exercise 1.11

1.13. $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\{0\}\}, A\}$

1.15. $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\emptyset\}, \{\{\emptyset\}\}, \{0, \emptyset\}, \{0, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, A\}$; $|\mathcal{P}(A)| = 8$

Section 1.3: Set Operations

1.17. **(a)** $A \cup B = \{1, 3, 5, 9, 13, 15\}$ **(b)** $A \cap B = \{9\}$ **(c)** $A - B = \{1, 5, 13\}$
(d) $B - A = \{3, 15\}$ **(e)** $\overline{A} = \{3, 7, 11, 15\}$ **(f)** $A \cap \overline{B} = \{1, 5, 13\}$

1.19. Let $A = \{1, 2\}$, $B = \{1, 3\}$, and $C = \{2, 3\}$. Then $B \neq C$ but $B - A = C - A = \{3\}$.

1.21. **(a)** and **(b)** are the same, as are **(c)** and **(d)**

1.23. See Figures 2(a) and (b) below.

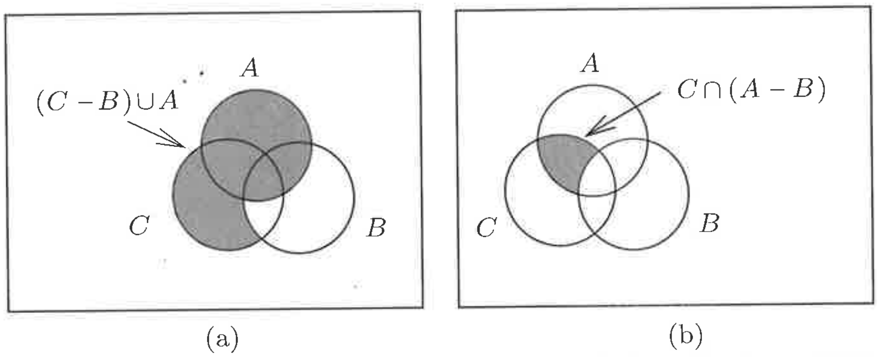

Figure 2 Answers for Exercise 1.23

Section 1.4: Indexed Collections of Sets

1.25. Let $U = \{1, 2, \ldots, 8\}$, $A = \{1, 2, 3, 5\}$, $B = \{1, 2, 4, 6\}$, and $C = \{1, 3, 4, 7\}$.

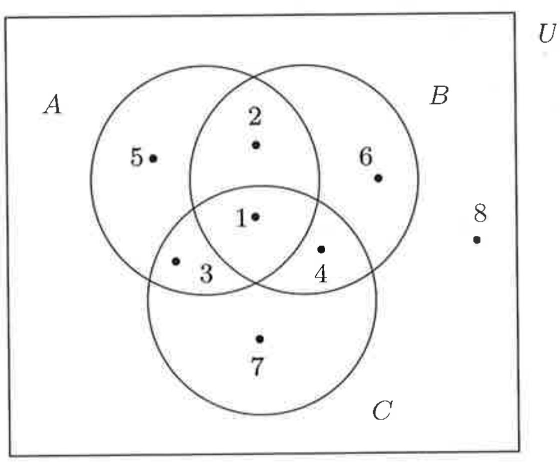

Figure 3 Answer for Exercise 1.25

1.27. $\bigcup_{X \in S} X = A \cup B \cup C = \{0, 1, 2, \ldots, 5\}$ and $\bigcap_{X \in S} X = A \cap B \cap C = \{2\}$.

1.29. Since $|A| = 26$ and $|A_\alpha| = 3$ for each $\alpha \in A$, we need to have at least nine sets of cardinality 3 for their union to be A; that is, in order for $\bigcup_{\alpha \in S} A_\alpha = A$, we must have $|S| \geq 9$. However, if we let $S = \{a, d, g, j, m, p, s, v, y\}$, then $\bigcup_{\alpha \in S} A_\alpha = A$. Hence the smallest cardinality of a set S with $\bigcup_{\alpha \in S} A_\alpha = A$ is 9.

1.31. **(a)** $\{A_n\}_{n \in \mathbf{N}}$, where $A_n = \{x \in \mathbf{R} : 0 \leq x \leq 1/n\} = [0, 1/n]$.
(b) $\{A_n\}_{n \in \mathbf{N}}$, where $A_n = \{a \in \mathbf{Z} : |a| \leq n\} = \{-n, -(n-1), \ldots, (n-1), n\}$.

Section 1.5: Partitions of Sets

1.33. **(a)** S_1 is not a partition of A since 4 belongs to no element of S_1.
(b) S_2 is a partition of A. S_2 can be written as $\{\{1, 2\}, \{3, 4, 5\}\}$.
(c) S_3 is not a partition of A because 2 belongs to two elements of S_3.
(d) S_4 is not a partition of A since S_4 is not a set of subsets of A.

1.35. $A = \{1, 2, 3, 4\}$. $S_1 = \{\{1\}, \{2\}, \{3, 4\}\}$ and $S_2 = \{\{1, 2\}, \{3\}, \{4\}\}$.

1.37. Let $S = \{A_1, A_2, A_3\}$, where $A_1 = \{x \in \mathbf{Q} : x > 1\}$, $A_2 = \{x \in \mathbf{Q} : x < 1\}$, and $A_3 = \{1\}$.

1.39. Let $S = \{A_1, A_2, A_3, A_4\}$, where $A_1 = \{x \in \mathbf{Z} : x \text{ is odd and } x \text{ is positive}\}$, $A_2 = \{x \in \mathbf{Z} : x \text{ is odd and } x \text{ is negative}\}$, $A_3 = \{x \in \mathbf{Z} : x \text{ is even and } x \text{ is nonnegative}\}$, $A_4 = \{x \in \mathbf{Z} : x \text{ is even and } x \text{ is negative}\}$.

Section 1.6: Cartesian Products of Sets

1.41. $A \times B = \{(x, x), (x, y), (y, x), (y, y), (z, x), (z, y)\}$.

1.43. $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, A\}$, $A \times \mathcal{P}(A) = \{(a, \emptyset), (a, \{a\}), (a, \{b\}), (a, A), (b, \emptyset), (b, \{a\}), (b, \{b\}), (b, A)\}$.

1.45. $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, A\}$, $\mathcal{P}(B) = \{\emptyset, B\}$, $A \times B = \{(1, \emptyset), (2, \emptyset)\}$,
$\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset), (\emptyset, B), (\{1\}, \emptyset), (\{1\}, B), (\{2\}, \emptyset), (\{2\}, B), (A, \emptyset), (A, B)\}$.

1.47. $S = \{(3, 0), (2, 1), (1, 2), (0, 3), (-3, 0), (-2, 1), (-1, 2), (2, -1), (1, -2), (0, -3), (-2, -1), (-1, -2)\}$.

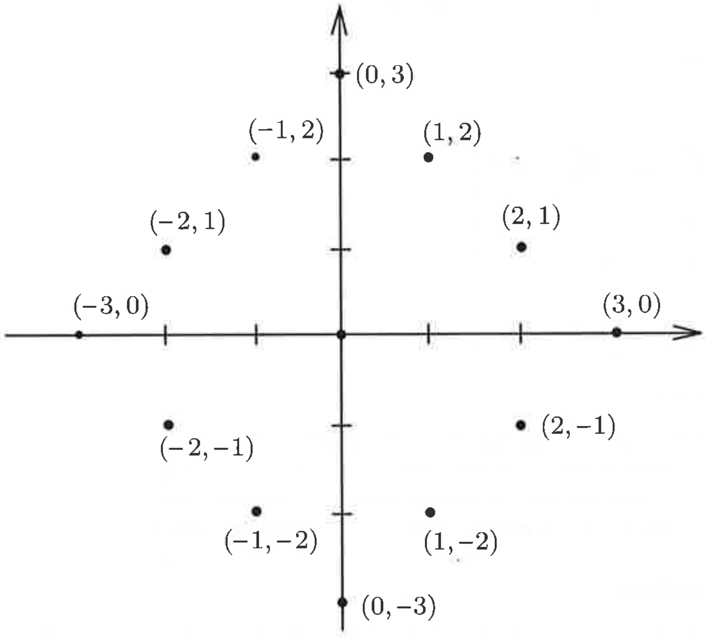

Figure 4 Answer for Exercise 1.47

EXERCISES FOR CHAPTER 2

Section 2.1: Statements

2.1. **(a)** A false statement **(b)** A true statement **(c)** Not a statement **(d)** Not a statement (an open sentence) **(e)** Not a statement **(f)** Not a statement (an open sentence) **(g)** Not a statement

2.3. **(a)** False. $\emptyset$ has no elements. **(b)** True **(c)** True
(d) False. $\{\emptyset\}$ has $\emptyset$ as its only element. **(e)** True **(f)** False. 1 is not a set.

2.5. **(a)** $\{x \in \mathbf{Z} : x > 2\}$ **(b)** $\{x \in \mathbf{Z} : x \leq 2\}$

2.7. 3, 5, 11, 17, 41, 59

Section 2.2: The Negation of a Statement

2.9. See Figure 5.

P	Q	$\sim P$	$\sim Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

Figure 5 Answer for Exercise 2.9

Section 2.3: The Disjunction and Conjunction of Statements

2.11. **(a)** True, **(b)** False, **(c)** False, **(d)** True, **(e)** True.

2.13. **(a)** All nonempty subsets of $\{1, 3, 5\}$. **(b)** All subsets of $\{1, 3, 5\}$.
(c) There are no subsets A of S for which $(\sim P(A)) \wedge (\sim Q(A))$ is true.

Section 2.4: The Implication

2.15. See Figure 6.

P	Q	$\sim P$	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow (\sim P)$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Figure 6 Answer for Exercise 2.15

2.17. **(a)** $(P \wedge Q) \Rightarrow R$: If $\sqrt{2}$ is rational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is rational. (True)
(b) $(P \wedge Q) \Rightarrow (\sim R)$: If $\sqrt{2}$ is rational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is not rational. (True)
(c) $((\sim P) \wedge Q) \Rightarrow R$: If $\sqrt{2}$ is not rational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is rational. (False)
(d) $(P \vee Q) \Rightarrow (\sim R)$: If $\sqrt{2}$ is rational or $\frac{2}{3}$ is rational, then $\sqrt{3}$ is not rational. (True)

Section 2.5: More On Implications

2.19. **(a)** $P(x) \Rightarrow Q(x)$: If $|x| = 4$, then $x = 4$. $P(-4) \Rightarrow Q(-4)$ is false. $P(-3) \Rightarrow Q(-3)$ is true. $P(1) \Rightarrow Q(1)$ is true. $P(4) \Rightarrow Q(4)$ is true. $P(5) \Rightarrow Q(5)$ is true.
(b) $P(x) \Rightarrow Q(x)$: If $x^2 = 16$, then $|x| = 4$. True for all $x \in S$.
(c) $P(x) \Rightarrow Q(x)$: If $x > 3$, then $4x - 1 > 12$. True for all $x \in S$.

2.21. **(a)** True for $(x, y) = (3, 4)$ and $(x, y) = (5, 5)$, false for $(x, y) = (1, -1)$.
(b) True for $(x, y) = (1, 2)$ and $(x, y) = (6, 6)$, false for $(x, y) = (2, -2)$.
(c) True for $(x, y) \in \{(1, -1), (-3, 4), (1, 0)\}$ and false for $(x, y) = (0, -1)$.

Section 2.6: The Biconditional

2.23. **(a)** $\sim P(x) : x \neq -2$. True if $x = 0, 2$.
(b) $P(x) \vee Q(x) : x = -2$ or $x^2 = 4$. True if $x = -2, 2$.
(c) $P(x) \wedge Q(x) : x = -2$ and $x^2 = 4$. True if $x = -2$.

(d) $P(x) \Rightarrow Q(x)$: If $x = -2$, then $x^2 = 4$. True for all x.
(e) $Q(x) \Rightarrow P(x)$: If $x^2 = 4$, then $x = -2$. True if $x = 0, -2$.
(f) $P(x) \Leftrightarrow Q(x)$: $x = -2$ if and only if $x^2 = 4$. True if $x = 0, -2$.

2.25. x is odd if and only if x^2 is odd.
That x is odd is a necessary and sufficient condition for x^2 to be odd.

2.27. **(a)** True for $(x, y) \in \{(3, 4), (5, 5)\}$. **(b)** True for $(x, y) \in \{(1, 2), (6, 6)\}$. **(c)** True for $(x, y) \in \{(1, -1), (1, 0)\}$.

2.29. **(i)** $P(1) \Rightarrow Q(1)$ is false; **(ii)** $Q(4) \Rightarrow P(4)$ is true;
(iii) $P(2) \Leftrightarrow R(2)$ is true; **(iv)** $Q(3) \Leftrightarrow R(3)$ is false.

Section 2.7: Tautologies and Contradictions

2.31. The compound statement $(P \wedge (\sim Q)) \wedge (P \wedge Q)$ is a contradiction since it is false for all combinations of truth values for the component statements P and Q. See the truth table below.

P	Q	$\sim Q$	$P \wedge Q$	$P \wedge (\sim Q)$	$(P \wedge (\sim Q)) \wedge (P \wedge Q)$
T	T	F	T	F	**F**
T	F	T	F	T	**F**
F	T	F	F	F	**F**
F	F	T	F	F	**F**

2.33. The compound statement $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a tautology since it is true for all combinations of truth values for the component statements P, Q, and R. See the truth table below.

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$P \Rightarrow R$	$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T	T	**T**
T	F	T	F	T	F	T	**T**
F	T	T	T	T	T	T	**T**
F	F	T	T	T	T	T	**T**
T	T	F	T	F	F	F	**T**
T	F	F	F	T	F	F	**T**
F	T	F	T	F	F	T	**T**
F	F	F	T	T	T	T	**T**

$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$: If P implies Q and Q implies R, then P implies R.

Section 2.8: Logical Equivalence

2.35. **(a)** See the truth table below.

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim(P \vee Q)$	$(\sim P) \vee (\sim Q)$
T	T	F	F	T	**F**	**F**
T	F	F	T	T	**F**	**T**
F	T	T	F	T	**F**	**T**
F	F	T	T	F	**T**	**T**

Since $\sim(P \vee Q)$ and $(\sim P) \vee (\sim Q)$ do not have the same truth values for all combinations of truth values for the component statements P and Q, the compound statements $\sim(P \vee Q)$ and $(\sim P) \vee (\sim Q)$ are not logically equivalent.

(b) The biconditional $\sim(P \vee Q) \Leftrightarrow ((\sim P) \vee (\sim Q))$ is not a tautology, and so there are instances when this biconditional is false.

2.37. The statements Q and $(\sim Q) \Rightarrow (P \wedge (\sim P))$ are logically equivalent since they have the same truth values for all combinations of truth values for the component statements P and Q. See the truth table below.

P	Q	$\sim P$	$\sim Q$	$P \wedge (\sim P)$	$(\sim Q) \Rightarrow (P \wedge (\sim P))$
T	**T**	F	F	F	**T**
T	**F**	F	T	F	**F**
F	**T**	T	F	F	**T**
F	**F**	T	T	F	**F**

Section 2.9: Some Fundamental Properties of Logical Equivalence

2.39. **(a)** The statement $P \vee (Q \wedge R)$ is equivalent to $(P \vee Q) \wedge (P \vee R)$ since the last two columns in the truth table of Figure 7 are the same.

P	Q	R	$P \vee Q$	$P \vee R$	$Q \wedge R$	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	**T**	**T**
T	F	T	T	T	F	**T**	**T**
F	T	T	T	T	T	**T**	**T**
F	F	T	F	T	F	**F**	**F**
T	T	F	T	T	F	**T**	**T**
T	F	F	T	T	F	**T**	**T**
F	T	F	T	F	F	**F**	**F**
F	F	F	F	F	F	**F**	**F**

Figure 7 Answer for Exercise 2.39(a)

(b) The statement $\sim(P \vee Q)$ is equivalent to $(\sim P) \wedge (\sim Q)$ since the last two columns in the truth table of Figure 8 are the same.

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim(P \vee Q)$	$(\sim P) \wedge (\sim Q)$
T	T	F	F	T	**F**	**F**
T	F	F	T	T	**F**	**F**
F	T	T	F	T	**F**	**F**
F	F	T	T	F	**T**	**T**

Figure 8 Answer for Exercise 2.39(b)

2.41. **(a)** x and y are even only if xy is even.
(b) If xy is even, then x and y are even.
(c) Either at least one of x and y is odd or xy is even.
(d) x and y are even and xy is odd.

Section 2.10: Quantified Statements

2.43. $\forall x \in S, P(x)$: For every odd integer x, the integer $x^2 + 1$ is even.
$\exists x \in S, Q(x)$: There exists an odd integer x such that x^2 is even.

2.45. **(a)** There exists a set A such that $A \cap \overline{A} \neq \emptyset$.
(b) For every set A, we have $\overline{A} \not\subseteq A$.

2.47. **(a)** False, since $P(1)$ is false. **(b)** True, for example, $P(3)$ is true.

2.49. **(a)** $\exists a, b \in \mathbf{Z}, ab < 0$ and $a + b > 0$.
(b) $\forall x, y \in \mathbf{R}, x \neq y$ implies that $x^2 + y^2 > 0$.
(c) For all integers a and b either $ab \geq 0$ or $a + b \leq 0$.
There exist real numbers x and y such that $x \neq y$ and $x^2 + y^2 \leq 0$.
(d) $\forall a, b \in \mathbf{Z}, ab \geq 0$ or $a + b \leq 0$.
$\exists x, y \in \mathbf{R}, x \neq y$ and $x^2 + y^2 \leq 0$.

2.51. Let $S = \{3, 5, 11\}$ and $P(s, t) : st - 2$ is prime.
(a) $\forall s, t \in S, P(s, t)$.
(b) True since $P(s, t)$ is true for all $s, t \in S$.
(c) $\exists s, t \in S, \sim P(s, t)$.
(d) There exist $s, t \in S$ such that $st - 2$ is not prime.
(e) False since the statement in (a) is true.

Section 2.11: Characterizations of Statements

2.53. An integer n is odd if and only if n^2 is odd.
2.55. **(a)** a characterization. **(b)** a characterization. **(c)** a characterization.
(d) a characterization. (Pythagorean theorem) **(e)** not a characterization. (Every positive number is the area of some rectangle.)

EXERCISES FOR CHAPTER 3

Section 3.1: Trivial and Vacuous Proofs

3.1. ***Proof*** Since $x^2 - 2x + 2 = (x - 1)^2 + 1 \geq 1$, it follows that $x^2 - 2x + 2 \neq 0$ for all $x \in \mathbf{R}$. Hence the statement is true trivially. ∎
3.3. ***Proof*** Note that $\frac{r^2+1}{r} = r + \frac{1}{r}$. If $r \geq 1$, then $r + \frac{1}{r} > 1$; while if $0 < r < 1$, then $\frac{1}{r} > 1$ and so $r + \frac{1}{r} > 1$. Thus $\frac{r^2+1}{r} \leq 1$ is false for all $r \in \mathbf{Q}^+$ and so the statement is true vacuously. ∎
3.5. ***Proof*** Since $n^2 - 2n + 1 = (n - 1)^2 \geq 0$, it follows that $n^2 + 1 \geq 2n$ and so $n + \frac{1}{n} \geq 2$. Thus the statement is true vacuously. ∎

Section 3.2: Direct Proofs

3.7. ***Proof*** Let x be an even integer. Then $x = 2a$ for some integer a. Thus

$$5x - 3 = 5(2a) - 3 = 10a - 4 + 1 = 2(5a - 2) + 1.$$

Since $5a - 2$ is an integer, $5x - 3$ is odd. ∎
3.9. ***Proof*** Let $1 - n^2 > 0$. Then $n = 0$. Thus $3n - 2 = 3 \cdot 0 - 2 = -2$ is an even integer. ∎
3.11. ***Proof*** Assume that $(n + 1)^2(n + 2)^2/4$ is even, where $n \in S$. Then $n = 2$. For $n = 2$, $(n + 2)^2(n + 3)^2/4 = 100$, which is even. ∎

Section 3.3: Proof by Contrapositive

3.13. First, we prove a lemma. **Lemma** Let $n \in \mathbf{Z}$. If $15n$ is even, then n is even.
(Use a proof by contrapositive to verify this lemma.) Then use this lemma to prove the result.
Proof of Result Assume that $15n$ is even. By the lemma, n is even and so $n = 2a$ for some integer a. Hence

$$9n = 9(2a) = 2(9a).$$

Since $9a$ is an integer, $9n$ is even. ∎
[Note: This result could also be proved by assuming that $15n$ is even (and so $15n = 2a$ for some integer a) and observing that $9n = 15n - 6n = 2a - 6n$.]
3.15. **Lemma** Let $x \in \mathbf{Z}$. If $7x + 4$ is even, then x is even. (Use a proof by contrapositive to verify this lemma.)
Proof of Result Assume that $7x + 4$ is even. Then by the lemma, x is even and so $x = 2a$ for some integer a. Hence

$$3x - 11 = 3(2a) - 11 = 6a - 12 + 1 = 2(3a - 6) + 1.$$

Since $3a - 6$ is an integer, $3x - 11$ is odd. ∎
3.17. The proof would begin by assuming that $n^2(n + 1)^2/4$ is odd, where $n \in S$. Then $n = 2$ and so $n^2(n - 1)^2/4 = 1$ is odd.

$2^t \notin S_k$. Thus $2^0, 2^1, \ldots, 2^{t-1} \in S_k$. Since $2^0 + 2^1 + \cdots + 2^{t-1} = 2^t - 1$, it follows that if we let

$$S_{k+1} = (S_k \cup \{2^t\}) - \{2^0, 2^1, \ldots, 2^{t-1}\},$$

then $\sum_{i \in S_{k+1}} i = k + 1 = m$, producing a contradiction. ∎

Section 6.4: The Strong Principle of Mathematical Induction

6.33. **Conjecture** A sequence $\{a_n\}$ is defined recursively by $a_1 = 1, a_2 = 2$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Then $a_n = 2^{n-1}$ for every positive integer n.

Proof We proceed by the Strong Principle of Mathematical Induction. Since $a_1 = 1$, the conjecture is true for $n = 1$. Assume that $a_i = 2^{i-1}$ for every integer i with $1 \leq i \leq k$, where $k \in \mathbf{N}$. We show that $a_{k+1} = 2^k$. Since $a_{1+1} = a_2 = 2 = 2^1$, it follows that $a_{k+1} = 2^k$ for $k = 1$. Hence we may assume that $k \geq 2$. Thus

$$\begin{aligned} a_{k+1} &= a_k + 2a_{k-1} = 2^{k-1} + 2 \cdot 2^{k-2} = 2^{k-1} + 2^{k-1} \\ &= 2 \cdot 2^{k-1} = 2^k. \end{aligned}$$

The result then follows by the Strong Principle of Mathematical Induction. ∎

6.35. **(a)** The sequence $\{F_n\}$ is defined recursively by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

(b) ***Proof*** We proceed by the Strong Principle of Mathematical Induction. Since $F_1 = 1$ is odd and $3 \nmid 1$, it follows that $2 \mid F_1$ if and only if $3 \mid 1$ and the statement is true for $n = 1$. Assume that $2 \mid F_i$ if and only if $3 \mid i$ for every integer i with $1 \leq i \leq k$ and $k \in \mathbf{N}$. We show that $2 \mid F_{k+1}$ if and only if $3 \mid (k+1)$. Since $F_2 = F_{1+1} = 1$ and $3 \nmid 2$, the statement is true for $k = 1$. Hence we may assume that $k \geq 2$. We now consider three cases, according to whether $k + 1 = 3q$, $k + 1 = 3q + 1$, or $k + 1 = 3q + 2$ for some integer q.

Case 1. $k + 1 = 3q$. Thus $3 \nmid k$ and $3 \nmid (k-1)$. By the inductive hypothesis, F_k and F_{k-1} are odd. Since $F_{k+1} = F_k + F_{k-1}$, it follows that F_{k+1} is even.

Case 2. $k + 1 = 3q + 1$. Thus $3 \mid k$ and $3 \nmid (k-1)$. By the inductive hypothesis, F_k is even and F_{k-1} is odd. Since $F_{k+1} = F_k + F_{k-1}$, it follows that F_{k+1} is odd.

Case 3. $k + 1 = 3q + 2$. Thus $3 \nmid k$ and $3 \mid (k-1)$. By the inductive hypothesis, F_k is odd and F_{k-1} is even. Since $F_{k+1} = F_k + F_{k-1}$, it follows that F_{k+1} is odd.

By the Strong Principle of Mathematical Induction, $2 \mid F_n$ if and only if $3 \mid n$ for every positive integer n. ∎

6.37. ***Proof*** We use the Strong Principle of Mathematical Induction. Since $12 = 3 \cdot 4 + 7 \cdot 0$, the statement is true when $n = 12$. Assume for an integer $k \geq 12$ that for every integer i with $12 \leq i \leq k$, there exist nonnegative integers a and b such that $i = 3a + 7b$. We show that there exist nonnegative integers x and y such that $k + 1 = 3x + 7y$. Since $13 = 3 \cdot 2 + 7 \cdot 1$ and $14 = 3 \cdot 0 + 7 \cdot 2$, we may assume that $k \geq 14$. Since $k - 2 \geq 12$, there exist nonnegative integers c and d such that $k - 2 = 3c + 7d$. Hence $k + 1 = 3(c+1) + 7d$. By the Strong Principle of Mathematical Induction, for each integer $n \geq 12$, there are nonnegative integers a and b such that $n = 3a + 7b$. ∎

EXERCISES FOR CHAPTER 7

Section 7.2: Revisiting Quantified Statements

7.1. **(a)** Let S be the set of all odd integers and let $P(n) : 3n + 1$ is even. $\forall n \in S, P(n)$.

(b) ***Proof*** Let $n \in S$. Then $n = 2k + 1$ for some integer k. Thus $3n + 1 = 3(2k + 1) + 1 = 6k + 4 = 2(3k + 2)$. Since $3k + 2$ is an integer, $3n + 1$ is even. ∎

7.3. **(a)** Let $P(n) : n^{n-1}$ is even. $\forall n \in \mathbf{N}, P(n)$.

(b) Note that $P(1)$ is false and so the statement in (a) is false.

7.5. **(a)** Let $P(m, n) : n < m < 2n$. $\forall n \in \mathbf{N} - \{1\}, \exists m \in \mathbf{Z}, P(m, n)$.

(b) ***Proof*** Let $n \geq 2$ be an integer and let $m = n + 1$. Since $n \geq 2$, it follows that $n < n + 1 = m < n + 2 \leq n + n = 2n$. ∎

7.7. **(a)** Let $P(m, n)$: $(n-2)(m-2) > 0$. $\forall n \in \mathbf{Z}, \exists m \in \mathbf{Z}, P(m, n)$.
(b) $\exists n \in \mathbf{Z}, \forall m \in \mathbf{Z}, \sim P(m, n)$.
(c) Let $n = 2$. Then $(n-2)(m-2) = 0 \cdot (m-2) = 0$ for all $m \in \mathbf{N}$.

7.9. **(a)** Let $P(a, b, x)$: $|bx| < a$ and $Q(a, b) : |b| < a$. $\forall a \in \mathbf{N}, \exists b \in \mathbf{Z}, (Q(a, b) \wedge (\forall x \in \mathbf{R}, P(a, b, x)))$.
(b) ***Proof*** Let $a \in \mathbf{N}$ and let $b = 0$. Then $|bx| = 0 < a$ for every real number x. ■

7.11. **(a)** Let $P(x, y, n)$: $x^2 + y^2 \geq n$. $\exists n \in \mathbf{Z}, \forall x, y \in \mathbf{R}, P(x, y, n)$.
(b) ***Proof*** Let $n = 0$. Then for every two real numbers x and y, $x^2 + y^2 \geq 0 = n$. ■

7.13. **(a)** Let $P(a, b, n)$: $a < \frac{1}{n} < b$. $\exists a, b \in \mathbf{Z}, \forall n \in \mathbf{N}, P(a, b, n)$.
(b) ***Proof*** Let $a = 0$ and $b = 2$. Then for every $n \in \mathbf{N}$, $a = 0 < \frac{1}{n} < 2 = b$. ■

7.15. **(a)** Let S be the set of odd integers and $P(a, b, c)$: abc is odd. $\forall a, b, c \in S, P(a, b, c)$.
(b) Let a, b, and c be odd integers. Then $a = 2x + 1$, $b = 2y + 1$, and $c = 2z + 1$, where $x, y, z \in \mathbf{Z}$. Then show that $abc = (2x + 1)(2y + 1)(2z + 1)$ is odd.

Section 7.3: Testing Statements

7.17. The statement is true.
Proof Since each of the following statements

$P(1) \Rightarrow Q(1)$: If 7 is prime, then 5 is prime.
$P(2) \Rightarrow Q(2)$: If 2 is prime, then 7 is prime.
$P(3) \Rightarrow Q(3)$: If 28 is prime, then 9 is prime.
$P(4) \Rightarrow Q(4)$: If 8 is prime, then 11 is prime.

is true, $\forall n \in S, P(n) \Rightarrow Q(n)$ is true. ■

7.19. This statement is false. Let $x = 1$. Then $4x + 7 = 11$ is odd and $x = 1$ is odd. Thus $x = 1$ is a counterexample.

7.21. This statement is true.
Proof Let x be an even integer. Then $x = 2n$ for some integer n. Observe that $x = (2n + 1) + (-1)$. Since n is an integer, $2n + 1$ is odd. Since -1 is odd as well, both $2n + 1$ and -1 are odd. ■

7.23. This statement is false. Let $A = \{1, 2, 3\}$ and $B = \{2, 3\}$. Then $A \cup B = \{1, 2, 3\}$ and $(A \cup B) - B = \{1\} \neq A$. Consequently, $A = \{1, 2, 3\}$ and $B = \{2, 3\}$ constitute a counterexample.

7.25. The statement is true.
Proof Consider the integer 35. Then $3 + 5 = 8$ is even and $3 \cdot 5 = 15$ is odd. ■

7.27. The statement is false. Let $x = 3$ and $y = -1$. Then $|x + y| = |3 + (-1)| = |2| = 2$ and $|x| + |y| = |3| + |-1| = 3 + 1 = 4$. Thus $|x + y| \neq |x| + |y|$. So $x = 3$ and $y = -1$ is a counterexample.

7.29. The statement is false. We show that there is no real number x such that $x^2 < x < x^3$.
Suppose that there is a real number x such that $x^2 < x < x^3$. Since $x^2 \geq 0$, it follows that $x > 0$. Dividing $x^2 < x < x^3$ by x, we have $x < 1 < x^2$. Thus $0 < x < 1$ and $x^2 > 1$, which is impossible. ■

7.31. The statement is true. For $a = 0$, any two real numbers b and $c \neq 0$ satisfy the equality.

7.33. The statement is false. Note that $x^4 + x^2 + 1 \geq 1 > 0$ for every $x \in \mathbf{R}$.

7.35. The statement is false. Neither of the expressions $\frac{x^3+x}{x^4-1}$ or $\frac{x}{x^2-1}$ is defined when $x = 1$ or $x = -1$.

7.37. The statement is false. Let $x = 6$ and $y = 4$. Then $z = 2$.

7.39. The statement is true.
Proof Assume that $A - B = \emptyset$ for every set B. Let $B = \emptyset$. Then $A - B = A - \emptyset = A = \emptyset$. ■

7.41. The statement is true.
Proof Let A be a nonempty set. Let $B = A$. Then $A - B = B - A = \emptyset$. So $|A - B| = |B - A| = 0$. ■

7.43. The statement is false. Observe that $4 = 1 + 3$.

7.45. The statement is true. Consider $c = 1$ and $d = 2b + 1$.

7.47. The statement is true. For each even integer n, $n = n + 0$.

7.49. The statement is false. Consider $A = \{1\}$, $B = \{2\}$, and $C = D = \{1, 2\}$.

7.51. The statement is true. Let $a = \sqrt{2}$ and $b = 1$.

7.53. The statement is true. Consider the set $B = S - A$.

7.55. The statement is false. Let $A = \{1\}$ and $B = \{2\}$. Then $\{1, 2\} \in \mathcal{P}(A \cup B)$ but $\{1, 2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

7.57. The statement is false. Consider $A = \{1\}$, $B = \{1, 2\}$, and $C = \{1\}$.

7.59. The statement is true. Observe that at least two of a, b, and c are of the same parity, say a and b are of the same parity. Then $a + b$ is even.

7.61. The statement is false. Consider $a = 2$ and $c = 1$.

7.63. The statement is false. Consider $n = 1$.

7.65. The statement is true. Let $x = 51$ and $y = 50$. Then $x^2 = (51)^2 = (50 + 1)^2 = (50)^2 + 2 \cdot 50 + 1$.

7.67. The statement is true.

Proof Let p be an odd prime. Then $p = 2k + 1$ for some $k \in \mathbf{N}$. For $a = k + 1$ and $b = k$, $a^2 - b^2 = (k + 1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1 = p$. ■

EXERCISES FOR CHAPTER 8

Section 8.1: Relations

8.1. $\text{dom}\, R = \{a, b\}$ and $\text{ran}\, R = \{s, t\}$.

8.3. Since $A \times A = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and $|A \times A| = 4$, the number of subsets of $A \times A$ and hence the number of relations on A is $2^4 = 16$. Four of these 16 relations are $\emptyset$, $A \times A$, $\{(0, 0)\}$, and $\{(0, 0), (0, 1), (1, 0)\}$.

Section 8.2: Properties of Relations

8.5. The relation R is reflexive and transitive. Since $(a, d) \in R$ and $(d, a) \notin R$, it follows that R is not symmetric.

8.7. The relation R is transitive but neither reflexive nor symmetric.

8.9. The relation R is reflexive and symmetric. Observe that $3\ R\ 1$ and $1\ R\ 0$ but $3 \not R\ 0$. Thus R is not transitive.

8.11. The relation R is symmetric and transitive but not reflexive.

8.13. The relation R is reflexive and symmetric. Observe that $-1\ R\ 0$ and $0\ R\ 2$ but $-1 \not R\ 2$. Thus R is not transitive.

Section 8.3: Equivalence Relations

8.15. ***Proof*** Since $a^3 = a^3$ for each $a \in \mathbf{Z}$, it follows that $a\ R\ a$ and R is reflexive. Let $a, b \in \mathbf{Z}$ such that $a\ R\ b$. Then $a^3 = b^3$ and so $b^3 = a^3$. Thus $b\ R\ a$ and R is symmetric. Let $a, b, c \in \mathbf{Z}$ such that $a\ R\ b$ and $b\ R\ c$. Thus $a^3 = b^3$ and $b^3 = c^3$. Hence $a^3 = c^3$ and so $a\ R\ c$ and R is transitive. ■

Let $a, b \in \mathbf{Z}$. Note that $a^3 = b^3$ if and only if $a = b$. Thus $[a] = \{a\}$ for every $a \in \mathbf{Z}$.

8.17. There are three distinct equivalence classes, namely $[1] = \{1, 5\}$, $[2] = \{2, 3, 6\}$, and $[4] = \{4\}$.

8.19. ***Proof*** Assume that $a\ R\ b$, $c\ R\ d$, and $a\ R\ d$. Since $a\ R\ b$ and R is symmetric, $b\ R\ a$. Similarly, $d\ R\ c$. Because $b\ R\ a$, $a\ R\ d$, and R is transitive, $b\ R\ d$. Finally, since $b\ R\ d$ and $d\ R\ c$, it follows that $b\ R\ c$, as desired. ■

Section 8.4: Properties of Equivalence Classes

8.21. Let $R = \{(v, v), (w, w), (x, x), (y, y), (z, z), (v, w), (w, v), (x, y), (y, x)\}$. Then $[v] = \{v, w\}$, $[x] = \{x, y\}$, and $[z] = \{z\}$ are three distinct equivalence classes.

8.23. Observe that $2\ R\ 6$ and $6\ R\ 3$, but $2 \not R\ 3$. Thus R is not transitive, and so R is not an equivalence relation.

8.25. ***Proof*** Let $x \in \mathbf{Z}$. Since $3x - 7x = -4x = 2(-2x)$ and $-2x$ is an integer, $3x - 7x$ is even. Thus $x\ R\ x$ and R is reflexive.

Next, we show that R is symmetric. Let $x\ R\ y$, where $x, y \in \mathbf{Z}$. Thus $3x - 7y$ is even and so $3x - 7y = 2a$ for some integer a. Observe that

$$3y - 7x = (3x - 7y) - 10x + 10y = 2a - 10x + 10y = 2(a - 5x + 5y).$$

Since $a - 5x + 5y$ is an integer, $3y - 7x$ is even. So $y\ R\ x$ and R is symmetric.

Finally, we show that R is transitive. Assume that $x\ R\ y$ and $y\ R\ z$, where $x, y, z \in \mathbf{Z}$. Then $3x - 7y$ and $3y - 7z$ are even. So $3x - 7y = 2a$ and $3y - 7z = 2b$, where $a, b \in \mathbf{Z}$. Adding these two equations, we obtain

$$(3x - 7y) + (3y - 7z) = 3x - 4y - 7z = 2a + 2b$$

and so $3x - 7z = 2a + 2b + 4y = 2(a + b + 2y)$. Since $a + b + 2y$ is an integer, $3x - 7z$ is even. Therefore, $x\ R\ z$ and R is transitive. ■

There are two distinct equivalence classes, namely, $[0] = \{0, \pm 2, \pm 4, \ldots\}$ and $[1] = \{\pm 1, \pm 3, \pm 5, \ldots\}$.