Homework 11 - Solutions

7.12 (a) We will prove that R is reflexive. For any $a \in \mathbb{Z}$, $a \cdot a \geq 0$, so $(a, a) \in R$. Thus R is reflexive.

(b) We will prove that R is symmetric. If $(a, b) \in R$, then $ab \geq 0$. Since $ba = ab$, $ba \geq 0$. So $(b, a) \in R$. Thus R is symmetric.

(c) We will show that R is not transitive. Consider $a = 1$, $b = 0$, and $c = -1$. Then $ab = 0 \geq 0$, $bc = 0 \geq 0$, and $ac = -1 \not\geq 0$. So $(a, b) \in R$, $(b, c) \in R$, but $(a, c) \notin R$. Thus R is not transitive.

7.14. R is reflexive because for any $a \in \mathbb{Z}$, $a^3 = a^3$, thus $(a, a) \in R$.

R is symmetric because if $(a, b) \in R$, then $a^3 = b^3$, then $b^3 = a^3$, thus $(b, a) \in R$.

R is transitive because if $(a, b) \in R$ and $(b, c) \in R$, then $a^3 = b^3$ and $b^3 = c^3$, then $a^3 = c^3$, thus $(a, c) \in R$.

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

To determine its equivalence classes, notice that $(a, b) \in R$ iff $a^3 = b^3$ iff $a = b$. Thus every integer is equivalent only to itself. This means that each element forms its own equivalence class, and thus there are infinitely many equivalence classes: $\ldots$, $[-1] = \{-1\}$, $[0] = \{0\}$, $[1] = \{1\}$, $[2] = \{2\}$, $\ldots$

7.16. Let's we compute the equivalence class of every element:

$[1] = \{x \mid (x, 1) \in R\} = \{1, 5\}$,

$[2] = \{x \mid (x, 2) \in R\} = \{2, 3, 6\}$,

$[3] = \{x \mid (x, 3) \in R\} = \{2, 3, 6\}$,

$[4] = \{x \mid (x, 4) \in R\} = \{4\}$,

$[5] = \{x \mid (x, 5) \in R\} = \{1, 5\}$,

$[6] = \{x \mid (x, 6) \in R\} = \{2, 3, 6\}$.

Thus there are three distinct equivalence classes: $\{1, 5\}$, $\{2, 3, 6\}$, and $\{4\}$.

7.20. Consider $R = \{(v, v), (v, w), (w, v), (w, w), (x, x), (x, y), (y, x), (y, y), (z, z)\}$. This equivalence relation has three distinct equivalence classes, namely, $\{v, w\}$, $\{x, y\}$, and $\{z\}$.

7.24. For any $x \in \mathbb{Z}$, $3x - 7x = -4x = 2(-2x)$. Since $-2x \in \mathbb{Z}$, $3x - 7x$ is even. So $(x, x) \in R$. Thus R is reflexive.

If $(x,y) \in R$, then $3x-7y$ is even, so $3x-7y=2k$ for some $k \in \mathbb{Z}$. Then $3y-7x = 3x-7y-10x+10y = 2k-10x+10y = 2(k-5x+5y)$. Since $k-5x+5y \in \mathbb{Z}$, $3y-7x$ is even. So $(y,x) \in R$. Thus R is symmetric.

If $(x,y) \in R$ and $(y,z) \in R$, then $3x-7y$ and $3y-7z$ are even. Then $3x-7y=2k$ and $3y-7z=2l$ for some $k,l \in \mathbb{Z}$. Then $3x-7z = 3x-7y+3y-7z+4y = 2k+2l+4y = 2(k+l+2y)$. Since $k+l+2y \in \mathbb{Z}$, $3x-7z$ is even. So $(x,z) \in R$. Thus R is transitive.

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

Next we determine the equivalence classes. Let's first compute $[0]$:

$[0] = \{x \mid (x,0) \in R\} = \{x \mid 3x \text{ is even}\} = \{x \mid x \text{ is even}\} = \{\ldots,-4,-2,0,2,4,\ldots\}$.

Since 1 is not in this equivalence class, next we will compute the equavalence class of 1:

$[1] = \{x \mid (x,1) \in R\} = \{x \mid 3x-7 \text{ is even}\} = \{x \mid 3x \text{ is odd}\} = \{x \mid x \text{ is odd}\} = \{\ldots,-3,-1,1,3,\ldots\}$.

Since $[0] \cup [1] = \mathbb{Z}$, these two classes are the only equivalence classes.

7.44. (a) (ii) Since $0 \cdot 0 = 0 \not> 0$, $(0,0) \notin R$, so R is not reflexive.
If $(a,b) \in R$, then $ab > 0$. Since $ba = ab$, $ba > 0$. So $(b,a) \in R$. Thus R is symmetric.
If $(a,b),(b,c) \in R$, then $ab > 0$ and $bc > 0$. It follows that a and b have the same sign and b and c have the same sign, i.e. all three of a, b, and c are of the same sign. Therefore $ac > 0$. Thus R is transitive.

(iv) An integer number is greater than or equal to 1 if and only if it is positive. Thus the conditions $xy \geq 1$ and $xy > 0$ are equivalent. Therefore the relation R in this part is exactly the same as the relation in part (ii), and thus has the same propoerties: not reflexive, symmetric, and transitive.

(v) Since $0 \cdot 0 = 0$ is not odd, $(0,0) \notin R$, so R is not reflexive.
If $(a,b) \in R$, then ab is odd. Since $ba = ab$, ba is odd, so $(b,a) \in R$. Thus R is symmetric.
If $(a,b),(b,c) \in R$, then ab and bc are both odd. It follows that all three of a, b, and c are odd. Therefore ac is odd. Thus R is transitive.