MATH 111 **Fall 2007**

Homework 12 - Solutions

7.28. (1) Since for any $a \in \mathbb{Z}$, $3a + 5a \equiv 8a \equiv 0 \pmod 8$, it follows that $(a, a) \in R$. Thus R is reflexive.

(2) If $(a, b) \in R$, then $3a+5b \equiv 0 \pmod 8$. Then $3b+5a \equiv (8a+8b)-(3a+5b) \equiv 0 \pmod 8$, so $(b, a) \in R$. Thus R is symmetric.

(3) If $(a, b) \in R$ and $(b, c) \in R$, then $3a+5b \equiv 0 \pmod 8$ and $3b+5c \equiv 0 \pmod 8$. Then $3a + 5c \equiv 3a + 8b + 5c \equiv (3a + 5b) + (3b + 5c) \equiv 0 \pmod 8$, so $(a, c) \in R$. Thus R is transitive.

7.34. (a) The statement is true. For example, if $a = 0$, then for any integer b, $ab = 0$. Since $3|0$, $ab \equiv 0 \pmod 3$.

(b) The statement is false. For example, if $a = 1$ and $b = 1$, then $ab = 1$, so $ab \not\equiv 0 \pmod 3$.

7.36. (1) Since for any $a \in \mathbb{Z}$, $5|(a^2 - a^2)$, it follows that $a^2 \equiv a^2 \pmod 5$, so $(a, a) \in R$. Thus R is reflexive.

(2) If $(a, b) \in R$, then $a^2 \equiv b^2 \pmod 5$. Then $b^2 \equiv a^2 \pmod 5$, so $(b, a) \in R$. Thus R is symmetric.

(3) If $(a, b) \in R$ and $(b, c) \in R$, then $a^2 \equiv b^2 \pmod 5$ and $b^2 \equiv c^2 \pmod 5$. Then $a^2 \equiv c^2 \pmod 5$, so $(a, c) \in R$. Thus R is transitive.

Since $0^2 \equiv 0 \pmod 5$, $1^2 \equiv 1 \pmod 5$, $2^2 \equiv 4 \pmod 5$, $3^2 \equiv 4 \pmod 5$, and $4^2 \equiv 1 \pmod 5$, the equivalence classes are:

$[0] = \{a \in Z \mid a^2 \equiv 0 \pmod 5\} = \{a \in Z \mid a \equiv 0 \pmod 5\} = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$,

$[1] = \{a \in Z \mid a^2 \equiv 1 \pmod 5\} = \{a \in Z \mid a \equiv 1 \pmod 5 \text{ or } a \equiv 4 \pmod 5\} = \{\ldots, -9, -6, -4, -1, 1, 4, 6, 9, \ldots\}$, and

$[2] = \{a \in Z \mid a^2 \equiv 4 \pmod 5\} = \{a \in Z \mid a \equiv 2 \pmod 5 \text{ or } a \equiv 3 \pmod 5\} = \{\ldots, -8, -7, -3, -2, 2, 3, 7, 8, \ldots\}$.

(Note: since the union of these three classes is $\mathbb{Z}$, there are no other classes.)

8.2. Let $R = \{(1, a), (1, b), (1, c)\}$. Then R is not a function because the image of 1 is not well-defined (and the images of 2 and 3 are not defined).

8.4. The set of all functions from A to B is $B^A = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$, where

$f_1 = \{(1, x), (2, x), (3, x)\}$, $f_2 = \{(1, x), (2, x), (3, y)\}$, $f_3 = \{(1, x), (2, y), (3, x)\}$, $f_4 = \{(1, x), (2, y), (3, y)\}$, $f_5 = \{(1, y), (2, x), (3, x)\}$, $f_6 = \{(1, y), (2, x), (3, y)\}$, $f_7 = \{(1, y), (2, y), (3, x)\}$, $f_8 = \{(1, y), (2, y), (3, y)\}$.

8.6. Let $f = \{(w,r),(x,r),(y,s),(z,s)\}$. Then f is not one-to-one because $w \neq x$ but $f(w) = f(x)$. Also, f is not onto because t is not in the image.

8.8. (a) If $f(n_1) = f(n_2)$, i.e. $2n_1 + 1 = 2n_2 + 1$, thne $2n_1 = 2n_2$, so $n_1 = n_2$. Thus f is injective.

(b) The function f is not surjective, because e.g. if $b = 2$, then the equation $2n + 1 = 2$ has no integer solutions (the only real solution is $n = 0.5$, but it is not an integer).

8.12. (a) The function f is not one-to-one because e.g. $-4 \neq 0$, but $f(-4) = 9 = f(0)$.

(b) The function f is not onto because for all $x \in \mathbb{R}$, $x^2+4x+9 = (x^2+4x+4)+5 = (x+2)^2 + 5 \geq 5$, so e.g. $b = 4$ is not in the image.

8.14. (a) Let $f(n) = n$. Then f is one-to-one (because $f(n_1) = f(n_2)$ implies $n_1 = n_2$) and onto (because for any $b \in \mathbb{N}$, let $a = b$, then $f(a) = a = b$).

(b) Let $f(n) = n + 1$. Then f is one-to-one (because $f(n_1) = f(n_2)$ implies $n_1 = n_2$) but not onto (because for any $n \in \mathbb{N}$, $f(n) = n + 1 \geq 2$, so $b = 1$ is not in the image).

(c) Let $f(n) = |n - 2| + 1$. Then f is not one-to-one (because $1 \neq 3$ but $f(1) = 2 = f(3)$) and onto (because for any $b \in \mathbb{N}$, let $a = b + 1$, then $f(a) = |a - 2| + 1 = |b - 1| + 1 = b - 1 + 1 = b$ since $b - 1 \geq 0$).

(d) Let $f(n) = 1$. Then f is not one-to-one (because e.g. $1 \neq 2$ but $f(1) = f(2)$) and not onto (because e.g. $b = 2$ is not in the image).