MATH 111 **Fall 2007**

Homework 13 - Solutions

8.19. First we will prove that f is injective. Let $f(a) = f(b)$. Then $\dfrac{5a+1}{a-2} = \dfrac{5b+1}{b-2}$. Therefore $(5a+1)(b-2) = (a-2)(5b+1)$, so $5ab-10a+b-2 = 10ab+a-10b-2$. It follows that $-10a+b = a-10b$, so $-11a = -11b$, thus $a = b$.

Next we will prove that f is surjective. For any $b \in \mathbb{R} - \{5\}$, let $a = \dfrac{2b+1}{b-5}$. Then $a \in \mathbb{R} - \{2\}$ (since $\dfrac{2b+1}{b-5} = 2$ is equivalent to $2b+1 = 2b-10$ which has no real solutions) and $f(a) = f\left(\dfrac{2b+1}{b-5}\right) = \dfrac{5 \cdot \frac{2b+1}{b-5} + 1}{\frac{2b+1}{b-5} - 2} = \dfrac{\frac{10b+5}{b-5} + 1}{\frac{2b+1}{b-5} - 2} = \dfrac{(10b+5)+(b-5)}{(2b+1-2(b-5)} = \dfrac{11b}{11} = b$, so b is in the image.

Since f is both injective and surjective, it is bijective.

8.22. The composition $g \circ f : A \to C$ is given by $g \circ f = \{(1,y),(2,x),(3,x),(4,x)\}$.

9.4. We will prove by Mathematical Induction.

Basis step: if $n = 1$, then $1 = 2 - 1$ is true.

Inductive step: assume that $1+5+9+\ldots+(4k-3) = 2k^2 - k$ for some $k \in \mathbb{N}$. We will prove that $1+5+9+\ldots+(4(k+1)-3) = 2(k+1)^2 - (k+1)$.

Observe that $1+5+9+\ldots+(4(k+1)-3) = (1+5+9+\ldots+(4k-3))+(4(k+1)-3) = (2k^2-k)+(4(k+1)-3) = 2k^2-k+4k+1 = 2k^2+3k+1 = (2k^2+4k+2)-(k+1) = 2(k^2+2k+1)-(k+1) = 2(k+1)^2-(k+1)$.

9.8(a). Using identity (9.8), we have $2^2+4^2+6^2+\ldots+(2n)^2 = (2\cdot 1)^2+(2\cdot 2)^2+(2\cdot 3)^2+\ldots+(2\cdot n)^2 = 2^2(1^2+2^2+3^2+\ldots+n^2) = 4\cdot\dfrac{n(n+1)(2n+1)}{6} = \dfrac{2n(n+1)(2n+1)}{3}$.

9.9. We will prove by Mathematical Induction.

Basis step: if $n = 1$, then $1 \cdot 2 = \dfrac{1\cdot 2\cdot 3}{3}$ is true.

Inductive step: assume that $1\cdot 2+2\cdot 3+3\cdot 4+\ldots+k(k+1) = \dfrac{k(k+1)(k+2)}{3}$ for some $k \in \mathbb{N}$. We will prove that $1\cdot 2+2\cdot 3+3\cdot 4+\ldots+(k+1)(k+2) = \dfrac{(k+1)(k+2)(k+3)}{3}$.

Observe that

$1\cdot2+2\cdot3+3\cdot4+\ldots+(k+1)(k+2) = (1\cdot2+2\cdot3+3\cdot4+\ldots+k(k+1))+(k+1)(k+2) = \dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \dfrac{k(k+1)(k+2)+3(k+1)(k+2)}{3} = \dfrac{(k+1)(k+2)(k+3)}{3}$.

9.12. We will prove by Mathematical Induction.

Basis step: if $n = 1$, then $\dfrac{1}{3\cdot4} = \dfrac{1}{3+9}$ is true.

Inductive step: assume that $\dfrac{1}{3\cdot4} + \dfrac{1}{4\cdot5} + \ldots + \dfrac{1}{(k+2)(k+3)} = \dfrac{k}{3k+9}$ for some $k \in \mathbb{N}$. We will prove that $\dfrac{1}{3\cdot4} + \dfrac{1}{4\cdot5} + \ldots + \dfrac{1}{(k+3)(k+4)} = \dfrac{k+1}{3k+12}$.

Observe that $\dfrac{1}{3\cdot4}+\dfrac{1}{4\cdot5}+\ldots+\dfrac{1}{(k+3)(k+4)} = \left(\dfrac{1}{3\cdot4} + \dfrac{1}{4\cdot5} + \ldots \dfrac{1}{(k+2)(k+3)}\right) + \dfrac{1}{(k+3)(k+4)} = \dfrac{k}{3k+9}+\dfrac{1}{(k+3)(k+4)} = \dfrac{k}{3(k+3)}+\dfrac{1}{(k+3)(k+4)} = \dfrac{k(k+4)+3}{3(k+3)(k+4)} = \dfrac{k^2+4k+3}{3(k+3)(k+4)} = \dfrac{(k+1)(k+3)}{3(k+3)(k+4)} = \dfrac{k+1}{3(k+4)} = \dfrac{k+1}{3k+12}$.