# Homework 3 - Solutions

2.14. (a) If five times $n$ plus three is prime, then seven times $n$ plus one is prime.
(b) If 13 is prime, then 15 is prime. This statement is false.
(c) If 33 is prime, then 43 is prime. This statement is true.

2.16. (a) The number $x$ is not equal to $-2$.
(b) Either $x$ is equal to $-2$ or $x^2$ is equal to 4.
(c) The number $x$ is equal to $-2$ and $x^2$ is equal to 4.
(d) If the number $x$ is equal to $-2$, then $x^2$ is equal to 4.
(e) If $x^2$ is equal to 4, then $x$ is equal to $-2$.
(f) The number $x$ is equal to $-2$ if and only if $x^2$ is equal to 4.

2.22. (a) We will verify logical equivalence using a truth table:

| $P$ | $Q$ | $R$ | $Q \wedge R$ | $P \vee (Q \wedge R)$ | $P \vee Q$ | $P \vee R$ | $(P \vee Q) \wedge (P \vee R)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | F | T | T | T | T |
| T | F | T | F | T | T | T | T |
| T | F | F | F | T | T | T | T |
| F | T | T | T | T | T | T | T |
| F | T | F | F | F | T | F | F |
| F | F | T | F | F | F | T | F |
| F | F | F | F | F | F | F | F |

Since $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ have the same truth values for all possible combinations of truth values of $P$, $Q$, and $R$, these compound propositions are logically equivalent.

(b) As in part (a), we will construct a truth table:

| $P$ | $Q$ | $P \vee Q$ | $\neg(P \vee Q)$ | $\neg P$ | $\neg Q$ | $(\neg P) \wedge (\neg Q)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | T | F | F | T | F |
| F | T | T | F | T | F | F |
| F | F | F | T | T | T | T |

Since $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$ have the same truth values for all possible combinations of truth values of $P$ and $Q$, these compound propositions are logically equivalent.

1. (a) True. For example, when $x = 0$, $\neg P(x)$ is true.
   (b) False. For example, when $x = 0$, both $P(x)$ and $Q(x)$ are false, so $P(x) \vee Q(x)$ is false.
   (c) True. For example, when $x = -2$, both $P(x)$ and $Q(x)$ are true, so $P(x) \wedge Q(x)$ is true.

(d) True. Whenever $P(x)$ is true, we have $x = -2$, and since $(-2)^2 = 4$, $Q(x)$ is true.

(e) True. For example, when $x = -2$, both $Q(x)$ and $P(x)$ are true, so the implication is true.

(f) False. For example, when $x = 2$, $P(x)$ is false and $Q(x)$ is true, so the biconditional is false.

2. (a) $\forall x F(Mike, x)$ (or $\forall y\ F(Mike, y)$, *etc.*; it doesn't matter which variable to use).

   (b) $\forall x \exists y F(x, y)$.

   (c) $\neg \exists x (F(x, Fred) \wedge F(x, Jerry))$.

3. The proposition $\exists y \forall x (x + y = 0)$ is false because for any value of $y$ if we consider $x = -y + 1$ then the equation $x + y = 0$ is not satisfied. The proposition $\forall x \exists y Q(x, y)$ is true because for any $x$ we can choose $y = -x$, then the equation $x + y = 0$ is satisfied.

4. (a) $\neg \forall y \exists x P(x, y) \equiv \exists y \neg \exists x P(x, y) \equiv \exists y \forall x \neg P(x, y)$

   (b) $\neg(\exists x \exists y \neg P(x, y) \wedge \forall x \forall y Q(x, y)) \equiv (\neg \exists x \exists y \neg P(x, y)) \vee (\neg \forall x \forall y Q(x, y)) \equiv$
   $(\forall x \neg \exists y \neg P(x, y)) \vee (\exists x \neg \forall y Q(x, y)) \equiv (\forall x \forall y \neg\neg P(x, y)) \vee (\exists x \exists y \neg Q(x, y)) \equiv$
   $(\forall x \forall y P(x, y)) \vee (\exists x \exists y \neg Q(x, y))$

5. (a) Propositions $\forall x (P(x) \Leftrightarrow Q(x))$ and $(\forall x P(x)) \Leftrightarrow (\forall x Q(x))$ are not logically equivalent.

   Let $P(x)$ denote "$x > 0$" and let $Q(x)$ denote "$x < 0$" where $x \in \mathbb{R}$. Then $\forall x (P(x) \Leftrightarrow Q(x))$ is false: for example, if $x = 1$, then $P(x)$ is true and $Q(x)$ is false. However, $(\forall x P(x)) \Leftrightarrow (\forall x Q(x))$ is true because both $\forall x P(x)$ and $\forall x Q(x)$ are false (e.g. $P(x)$ is false for $x = -1$ and $Q(x)$ is false for $x = 1$).

   (b) Propositions $\exists x (P(x) \Leftrightarrow Q(x))$ and $(\exists x P(x)) \Leftrightarrow (\exists x Q(x))$ are not logically equivalent.

   Let $P(x)$ denote "$x < 0$" and let $Q(x)$ denote "$x^2 < 0$". Then $\exists x (P(x) \Leftrightarrow Q(x))$ is true: for example, if $x = 1$, then both $P(x)$ and $Q(x)$ are false. However, $(\exists x P(x)) \Leftrightarrow (\exists x Q(x))$ is false because $\exists x P(x)$ is true (e.g. for $x = -1$) and $\exists x Q(x)$ is false (since the square of any real number is nonnegative).

Note: problems 1, 3, and 5 admit other correct explanations/examples.