Homework 6 - Solutions

4.6. We will prove this by contrapositive. If $3 \not| a$, then either $a = 3k+1$ or $a = 3k+2$ for some $k \in \mathbb{Z}$.

Case I: $a = 3k+1$ for some $k \in \mathbb{Z}$. Then $2a = 2(3k++1) = 3(2a)+2$, therefore $3 \not| 2a$.

Case II: $a = 3k+2$ for some $k \in \mathbb{Z}$. Then $2a = 2(3k+2) = 3(2k+1)+1$, therefore $3 \not| 2a$.

4.8. If $a \equiv b \pmod{n}$, then $n|(a-b)$. Therefore $n|((a-b)(a+b))$, i.e. $n|(a^2-b^2)$. Thus $a^2 \equiv b^2 \pmod{n}$.

4.10. We will prove the statement by contrapositive. Assume that it is not the case that either a and b are both congruent to 0 modulo 3 or neither is congruent to 0 modulo 3. Then exactly one of a, b is congruent to 0 modulo 3. We have four cases.

Case I: $a \equiv 0 \pmod{3}$ and $b \equiv 1 \pmod{3}$. Then $a^2+2b^2 \equiv 0^2+2\cdot 1^2 \equiv 2 \pmod{3}$, thus $a^2+2b^2 \not\equiv 0 \pmod{3}$.

Case II: $a \equiv 0 \pmod{3}$ and $b \equiv 2 \pmod{3}$. Then $a^2+2b^2 \equiv 0^2+2\cdot 2^2 \equiv 8 \equiv 2 \pmod{3}$, thus $a^2+2b^2 \not\equiv 0 \pmod{3}$.

Case III: $a \equiv 1 \pmod{3}$ and $b \equiv 0 \pmod{3}$. Then $a^2+2b^2 \equiv 1^2+2\cdot 0^2 \equiv 1 \pmod{3}$, thus $a^2+2b^2 \not\equiv 0 \pmod{3}$.

Case IV: $a \equiv 2 \pmod{3}$ and $b \equiv 0 \pmod{3}$. Then $a^2+2b^2 \equiv 2^2+2\cdot 0^2 \equiv 4 \equiv 1 \pmod{3}$, thus $a^2+2b^2 \not\equiv 0 \pmod{3}$.

4.12. (a) The converse: Let $n \in \mathbb{Z}$. If $n \not\equiv 0 \pmod{3}$ and $n \not\equiv 1 \pmod{3}$, then $n^2 \not\equiv n \pmod{3}$.

Proof. The conjunction $n \not\equiv 0 \pmod{3}$ and $n \not\equiv 1 \pmod{3}$ implies $n \equiv 2 \pmod{3}$. Then $n^2 \equiv 4 \pmod{3}$. Since $4 \not\equiv 2 \pmod{3}$, it follows that $n^2 \not\equiv n \pmod{3}$.

(b) Let $n \in \mathbb{Z}$. Then $n^2 \not\equiv n \pmod{3}$ if and only if $n \not\equiv 0 \pmod{3}$ and $n \not\equiv 1 \pmod{3}$.

4.42. The proof contains a mistake. In the third sentence they assume that $x-1 = 3q$ and $y-1 = 3q$ for some integer q, which implies that $x = y$. However, x and y may not be equal, so different variables must be used, e.g. $x-1 = 3q$ and $y-1 = 3r$ for some integers q and r.