Homework 7 - Solutions

4.14. Proof by contrapositive. If $a < 3m + 1$ and $b < 2m + 1$, then since a, $3m + 1$, b, and $2m + 1$ are integers, it follows that $a \leq 3m$ and $b \leq 2m$. Then $2a + 3b \leq 2 \cdot 3m + 3 \cdot 2m = 12m < 12m + 1$, so the inequality $2a + 3b \geq 12m + 1$ does not hold. Therefore $2a + 3b \geq 12m + 1$ implies $a \geq 3m + 1$ or $b \geq 2m + 1$.

4.18. Each of x and y is either nonnegative or negative. Thus we will consider the following cases.

Case I: $x \geq 0$, $y \geq 0$. Then $xy \geq 0$. Therefore $|xy| = xy = |x| \cdot |y|$.

Case II: $x < 0$, $y < 0$. Then $xy > 0$. Therefore $|xy| = xy = (-x)(-y) = |x| \cdot |y|$.

Case III: one of x and y is nonnegative and the other one is positive. Without loss of generality we can assume that $x \geq 0$ and $y < 0$. Then $xy \leq 0$. Therefore $|xy| = -(xy) = x(-y) = |x| \cdot |y|$.

4.20. First we will show that $A \cup B \subset (A - B) \cup (B - A) \cup (A \cap B)$.
Let $x \in A \cup B$. Then $x \in A$ or $x \in B$ (or both). We will consider three cases:

Case I: $x \in A$ and $x \notin B$. Then $x \in A - B$, therefore $x \in (A-B) \cup (B-A) \cup (A \cap B)$.

Case II: $x \in B$ and $x \notin A$. Then $x \in B - A$, therefore $x \in (A-B) \cup (B-A) \cup (A \cap B)$.

Case III: $x \in A$ and $x \in B$. Then $x \in A \cap B$, therefore $x \in (A - B) \cup (B - A) \cup (A \cap B)$.

Next we will show that $(A - B) \cup (B - A) \cup (A \cap B) \subset A \cup B$.
Let $x \in (A - B) \cup (B - A) \cup (A \cap B)$. Then $x \in (A - B)$ or $x \in (B - A)$ or $x \in (A \cap B)$. So we will consider these three cases.

Case I: $x \in (A - B)$. Then $x \in A$, therefore $x \in A \cup B$.

Case II: $x \in (B - A)$. Then $x \in B$, therefore $x \in A \cup B$.

Case III: $x \in (A \cap B)$. Then $x \in A$, therefore $x \in A \cup B$.

4.22. First we will prove that if $A \cap B = A$, then $A \subset B$.
Let $x \in A$. Since $A \cap B = A$, $x \in A \cap B$. Therefore $x \in B$, so $A \subset B$.

Next we will prove that if $A \subset B$, then $A \cap B = A$.
To show $A \cap B = A$, we have to show that $A \cap B \subset A$ and $A \subset A \cap B$. The first inclusion holds because if $x \in A \cap B$, then $x \in A$. To show the second inclusion, let $x \in A$. Since $A \subset B$, by definition $x \in B$. Then $x \in A \cap B$.

4.23. (b) Let $A = \{1, 2\}$, $B = \emptyset$, $C = \{1\}$. Then $A \cup B = \{1, 2\}$ and $A \cup C = \{1, 2\}$, but $B \neq C$.

(c) Assume that $A \cap B = A \cap C$ and $A \cup B = A \cup C$. First we will show that $B \subset C$. Let $x \in B$. Since either $x \in A$ or $x \notin A$, we will consider two cases.

Case I: $x \in A$. Then $x \in A \cap B$, therefore $x \in A \cap C$. It follows that $x \in C$.

Case II: $x \notin A$. Since $x \in B$, $x \in A \cup B$. Therefore $x \in A \cup C$. Then $x \in A$ or $x \in C$. Since $x \notin A$, it follows that $x \in C$.

The proof of $C \subset B$ is similar.

4.24. Let $A \cup B \neq \emptyset$. Then there exists an element $x \in A \cup B$. By definition, $x \in A$ or $x \in B$. If $x \in A$, then $A \neq \emptyset$. If $x \in B$, then $B \neq \emptyset$.

4.27. First we will show that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.
Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in (B \cup C)$. Than $x \in B$ or $x \in C$. If $x \in B$, then $x \in (A \cap B)$, and therefore $x \in (A \cap B) \cup (A \cap C)$. If $x \in C$, then $x \in (A \cap C)$, and therefore $x \in (A \cap B) \cup (A \cap C)$.

Next we will show that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.
Let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in (A \cap B)$ or $x \in (A \cap C)$. If $x \in (A \cap B)$, then $x \in A$ and $x \in B$, therefore $x \in (B \cup C)$, and thus $x \in A \cap (B \cup C)$. If $x \in (A \cap C)$, then $x \in A$ and $x \in C$ therefore $x \in (B \cup C)$, and thus $x \in A \cap (B \cup C)$.

4.28. First we will show that $\overline{A \cap B} \subset \overline{A} \cup \overline{B}$.
Let $x \in \overline{A \cap B}$. Then $x \notin A \cap B$. Therefore $x \notin A$ or $x \notin B$, i.e. $x \in \overline{A}$ or $x \in \overline{B}$. It follows that $x \in \overline{A} \cup \overline{B}$.

Next we will show that $\overline{A} \cup \overline{B} \subset \overline{A \cap B}$.
Let $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$, so $x \notin A$ or $x \notin B$. Since x is not in both A and B, $x \notin A \cap B$. Therefore $x \in \overline{A \cap B}$.

4.37. First we will show that $(A \times B) \cap (C \times D) \subset (A \cap C) \times (B \cap D)$.
Let $x \in (A \times B) \cap (C \times D)$. Then $x \in (A \times B)$ and $x \in (C \times D)$. Therefore $x = (y, z)$ where $y \in A$, $z \in B$, $y \in C$, and $z \in D$. This implies that $y \in (A \cap C)$ and $z \in (B \cap D)$, thus $x = (y, z) \in (A \cap C) \times (B \cap D)$.

Next we will show that $(A \cap C) \times (B \cap D) \subset (A \times B) \cap (C \times D)$.
Let $x \in (A \cap C) \times (B \cap D)$. Then $x = (y, z)$ where $y \in (A \cap C)$ and $z \in (B \cap D)$. Therefore $y \in A$, $y \in C$, $z \in B$, and $z \in D$. This implies that $(y, z) \in (A \times B)$ and $(y, z) \in (C \times D)$, thus $x = (y, z) \in (A \times B) \cap (C \times D)$.