Homework 9 - Solutions

6.4. The statement is true. Consider $a = 0$, $b = 1$, and $c = 2$. Then $\dfrac{a+b}{a+c} = \dfrac{1}{2} = \dfrac{b}{c}$.

6.6. The statement is false. If $x \in \mathbb{R}$, then $x^2 \geq 0$ and $x^4 \geq 0$, so $x^4 + x^2 + 1 \geq 0 + 0 + 1 = 1 > 0$, thus $x^4 + x^2 + 1 \neq 0$.

6.10. The statement is false. Consider $x = y = z = 0$. Then $z = x - y$ and z is even, but x and y are not odd.

6.12. The statement is true. For any positive rational number b, let $a = \dfrac{b}{\sqrt{2}}$. Since $\sqrt{2} > 1$, $0 < a < b$. Next we will prove (by contradiction) that a is irrational. Assume a is rational. Then $a = \dfrac{k}{l}$ and $b = \dfrac{m}{n}$ for some $k, l, m, n \in \mathbb{Z}$, $l \neq 0$, $n \neq 0$. Since both a and b are positive, we also have $k \neq 0$ and $m \neq 0$. Then $\dfrac{k}{l} = \dfrac{m}{n\sqrt{2}}$. Therefore $\sqrt{2} = \dfrac{lm}{kn}$. Since $lm, kn \in \mathbb{Z}$ and $kn \neq 0$, $\sqrt{2}$ is rational. We get a contradiction.

6.14. The statement is true. Let $n \in \mathbb{Z}$ be odd. Then $n = 1 + (-1) + n$ shows that n is the sum of three odd integers.

6.16. The statement is false. For example, if $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 4\}$, then $A \cap B = \{1\} = A \cap C$ but $B \neq C$.

6.18. The statement is true. For any two rational numbers a and b with $a < b$, let $r = \dfrac{a+b}{2}$. Since a and b are rational, $a = \dfrac{k}{l}$ and $b = \dfrac{m}{n}$ for some $k, l, m, n \in \mathbb{Z}$, $l \neq 0$, $n \neq 0$. Then $r = \dfrac{a+b}{2} = \dfrac{\frac{k}{l} + \frac{m}{n}}{2} = \dfrac{kn + ml}{2ln}$. Since $kn + lm, 2ln \in \mathbb{Z}$ and $2ln \neq 0$, r is rational. Also, $a = \dfrac{a+a}{2} < \dfrac{a+b}{2} < \dfrac{b+b}{2} = b$, so $a < r < b$.

6.33. The statement is true. We will consider two cases.

Case I: at least two of a, b, c are even, say, a and b are even. Then $a = 2k$, $b = 2l$ for some $k, l \in \mathbb{Z}$. Then $a + b = 2k + 2l = 2(k + l)$. Since $k + l \in \mathbb{Z}$, $a + b$ is even.

Case II: at most one of a, b, c is even, then at least two of them are odd, say, a and b are odd. Then $a = 2k+1$, $b = 2l+1$ for some $k, l \in \mathbb{Z}$. Then $a+b = 2k+1+2l+1 = 2(k + l + 1)$. Since $k + l + 1 \in \mathbb{Z}$, $a + b$ is even.

Note 1: Problems 4, 10, 12, 14, 16, and 18 admit other examples. Giving one specific example is the best approach to all of these.
Note 2: For problem 6.33, a more straightforward (but a bit longer) approach is to consider four cases: all three numbers are even; two are even and one is odd; one is even and two are odd; all three are odd.