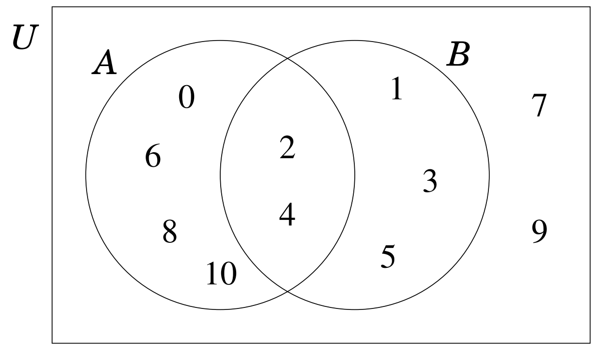
# Practice Test 1 - Solutions

1. Read the textbook!
2. (a)




   (b) $A \cap B = \{2, 4\}$, $\overline{A} = \{1, 3, 5, 7, 9\}$, $A \cup \overline{B} = \{0, 2, 4, 6, 7, 8, 9, 10\}$

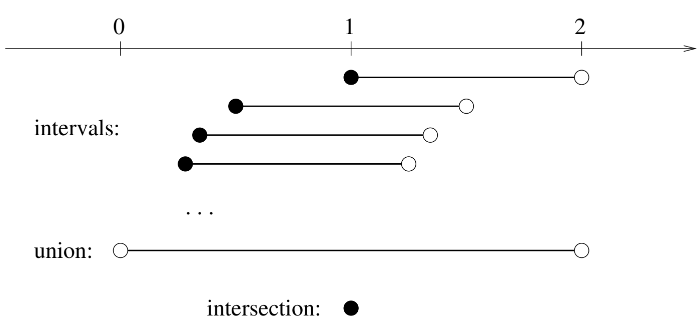
   (c) Since $A$ has six elements and $B$ has five elements, $A \times B$ has $6 \cdot 5 = 30$ elements.

   (d) $(0, 1)$, $(0, 2)$, $(10, 5)$.

3. (a) Statements $A \subset D$, $B \in D$, $\emptyset \subset D$ are true. The other statements are false.

   (b) $|A| = |B| = |C| = 1$, $|D| = 3$.

4. First of all, let's rewrite the right endpoint: $A_n = \left[\frac{1}{n}, 1 + \frac{1}{n}\right)$. Then the first few intervals are: $A_1 = [1, 2)$, $A_2 = \left[\frac{1}{2}, 1 + \frac{1}{2}\right)$, $A_3 = \left[\frac{1}{3}, 1 + \frac{1}{3}\right)$, etc. We see that the left endpoint approaches 0 and the right endpoint approaches 1 as $n$ gets larger:




   Therefore the union of these intervals is $\cup_{n \in \mathbb{N}} A_n = (0, 2)$ and the intersection is $\cap_{n \in \mathbb{N}} A_n = \{1\}$.

5. (a) We will use a truth table to show that $P \Leftrightarrow Q$ and $(P \wedge Q) \vee ((\neg P) \wedge (\neg Q))$ are logically equivalent.

| $P$ | $Q$ | $P \Leftrightarrow Q$ | $P \wedge Q$ | $\neg P$ | $\neg Q$ | $(\neg P) \wedge (\neg Q)$ | $(P \wedge Q) \vee ((\neg P) \wedge (\neg Q))$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | F | F | F | T |
| T | F | F | F | F | T | F | F |
| F | T | F | F | T | F | F | F |
| F | F | T | F | T | T | T | T |

Since the truth values of $P \Leftrightarrow Q$ and $(P \wedge Q) \vee ((\neg P) \wedge (\neg Q))$ are the same for all possible combinations of truth values of $P$ and $Q$, these compound propositions are logically equivalent.

(b) The compound statement $(P \Leftrightarrow Q) \Leftrightarrow ((P \wedge Q) \vee ((\neg P) \wedge (\neg Q)))$ is a <u>tautology</u>.

(c) The compound statement $(P \Leftrightarrow Q) \Leftrightarrow \neg((P \wedge Q) \vee ((\neg P) \wedge (\neg Q)))$ is a <u>contradiction</u>.

6. (a) $\exists! x\ (x^2 = 8)$ is false: there are two values of $x$ that satisfy $x^2 = 8$, namely, $\sqrt{8}$ and $-\sqrt{8}$.

(b) $\forall x \exists y\ (xy = 0)$ is true: for any $x$ we can choose $y = 0$, then we have $xy = 0$.

(c) $\forall x \exists! y\ (xy = 0)$ is false: if $x = 0$, then the value of $y$ is not unique, e.g. $y = 1$ and $y = 2$ satisfy $xy = 0$.

(d) $\exists x \forall y\ (xy = 0)$ is true: let $x = 0$, then for any $y$ we have $xy = 0$.

(e) $\exists! x \forall y\ (xy = 0)$ is true: if $x = 0$, then for any $y$ we have $xy = 0$. Also, this is the only value of $x$ such that for any $y$ the equation $xy = 0$ is satisfied, because if $x \neq 0$, then e.g. for $y = 1$ the equation $xy = 0$ is not satisfied.

(f) $\forall x \forall z \exists y\ (x + y = z)$ is true: for any $x$ and for any $z$ we can choose $y = z - x$, and then we have $x + y = z$.

(g) $\forall x \exists y \forall z\ (x + y = z)$ is false: given $x$, no matter what $y$ we choose, the value $z = x + y + 1$ does not satisfy $x + y = z$.

7. In all examples below, let $x$ and $y$ be real numbers.

(a)
- $\exists x \exists y P(x, y)$ is true if $P(x, y)$ is "$x + y = 0$" (e.g., let $x = 0$ and $y = 0$);
- $\exists x \exists y P(x, y)$ is false if $P(x, y)$ is "$x^2 + y^2 = -1$" (there are no values of $x$ and $y$ that satisfy the equation because the square of any real number is nonnegative).

(b)
- $\exists x \forall y P(x, y)$ is true if $P(x, y)$ is "$xy = 0$" (see problem 6(d));
- $\exists x \forall y P(x, y)$ is false if $P(x, y)$ is "$x + y = 0$" (no matter what $x$ is, the value $y = -x + 1$ does not satisfy the equation $x + y = 0$).

(c)
- $\forall x \exists y P(x, y)$ is true if $P(x, y)$ is "$xy = 0$" (see problem 6(b));
- $\forall x \exists y P(x, y)$ is false if $P(x, y)$ is "$xy = 1$" (if $x = 0$, there is no value of $y$ that satisfies the equation $xy = 1$).

(d) 
- $\forall x \forall y P(x, y)$ is true if $P(x, y)$ is "$x^2 + y^2 \geq 0$" (any real number squared is nonnegative, so the left hand side is nonnegative);
- $\forall x \forall y P(x, y)$ is false if $P(x, y)$ is "$x + y = 0$" (if $x = 1$ and $y = 2$, the equation is not satisfied).

8. (a) We will show that for any integer $n$, the number $3n^2 + 5n$ is even. To do this, we will consider two cases:
Case I: $n$ is even. Then $n = 2k$ for some $k \in \mathbb{Z}$, and $3n^2 + 5n = 3(2k)^2 + 5(2k) = 12k^2 + 10k = 2(6k^2 + 5k)$. Since $6k^2 + 5k \in \mathbb{Z}$, the number $3n^2 + 5n$ is even.
Case II: $n$ is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$, and $3n^2 + 5n = 3(2k + 1)^2 + 5(2k+1) = 12k^2 + 12k + 3 + 10k + 5 = 12k^2 + 22k + 8 = 2(6k^2 + 11k + 4)$. Since $6k^2 + 11k + 4 \in \mathbb{Z}$, the number $3n^2 + 5n$ is even.
Since $3n^2 + 5n$ is never odd, the implication follows. (This is a vacuous proof.)

(b) If $n$ is even, then $n = 2k$ for some $k \in \mathbb{Z}$, and $3n^2 - 2n - 5 = 3(2k)^2 - 2(2k) - 5 = 12k^2 - 4k - 5 = 12k^2 - 4k - 6 + 1 = 2(6k^2 - 2k - 3) + 1$. Since $6k^2 - 2k - 3 \in \mathbb{Z}$, the number $3n^2 - 2n - 5$ is odd. (This is a direct proof.)

(c) We will prove the statement by contrapositive, namely, we will prove that if $n$ and $m$ are of the same parity, then $n - 5m$ is even, and thus not odd.
Let's consider two cases:
Case I: $n$ and $m$ are both even. Then $n = 2k$ and $m = 2l$ for some $k, l \in \mathbb{Z}$. Then $n - 5m = 2k - 5(2l) = 2k - 10l = 2(k - 5l)$. Since $k - 5l \in \mathbb{Z}$, the number $n - 5m$ is even.
Case II: $n$ and $m$ are both odd. Then $n = 2k + 1$ and $m = 2l + 1$ for some $k, l \in \mathbb{Z}$. Then $n - 5m = 2k + 1 - 5(2l + 1) = 2k + 1 - 10l - 5 = 2k - 10l - 4 = 2(k - 5l - 2)$. Since $k - 5l - 2 \in \mathbb{Z}$, the number $n - 5m$ is even.

9. (a) For any real number $x$, $x^2 \geq 0$, therefore $-x^2 \leq 0$, and $-5 - x^2 \leq -5 + 0 = -5 < 0$. (This is a trivial proof.)

(b) If $|x| = 5$, then either $x = 5$ or $x = -5$. Thus we can consider the following two cases:
Case I: $x = 5$. Then $x^2 + x + 1 = 5^2 + 5 + 1 = 31 > 20$.
Case II: $x = -5$. Then $x^2 + x + 1 = (-5)^2 + (-5) + 1 = 21 > 20$.
(This is a proof by cases.)