MATH 111 **Fall 2007**

Practice Test 3 - Solutions

1. Read the textbook.
2. (a) $\{(a, 1), (b, 2), (c, 3)\}$ is a relation from B to A (since it is a subset of $B \times A$). Moreover, it is a function from B to A (since each element of B is the first coordinate of exactly one pair in the relation).

   (b) $\{(1, b), (1, c), (3, a), (4, b)\}$ is a relation from A to B (since it is a subset of $A \times B$), but it is not a function (e.g. since the image of 1 is not well-defined).
3. (a) The relation R is not reflexive: e.g. $(1, 1) \notin R$ since $1 + 1 \neq 0$;
   R is symmetric since if $(a, b) \in R$, then $a+b = 0$, then $b+a = 0$, so $(b, a) \in R$;
   R is not transitive: e.g. $(1, -1) \in R$ and $(-1, 1) \in R$, but $(1, 1) \notin R$;
   R is not an equivalence relation: e.g. R is not reflexive.

   (b) The relation R is not reflexive: e.g. $(0, 0) \notin R$ since $\frac{0}{0}$ is undefined, so it is not an element of $\mathbb{Q}$;
   R is not symmetric: e.g. $(0, 1) \in R$ since $\frac{0}{1} \in \mathbb{Q}$, but $(1, 0) \notin R$ since $\frac{1}{0}$ is undefined;
   R is transitive since if $(a, b) \in R$ and $(b, c) \in R$, then $\frac{a}{b} \in \mathbb{Q}$ and $\frac{b}{c} \in \mathbb{Q}$, and then $\frac{a}{c} \in \mathbb{Q}$;
   R is not an equivalence relation: e.g. R is not reflexive.

   (c) The relation R is not reflexive: $(0, 0) \notin R$ since $0 \cdot 0 \not> 0$;
   R is symmetric since if $(a, b) \in R$, then $ab > 0$, then $ba > 0$, so $(b, a) \in R$;
   R is transitive since if $(a, b) \in R$ and $(b, c) \in R$, then $ab > 0$ and $bc > 0$, then either all of a, b, and c are positive or all of them are negative; in either case, $ac > 0$, so $(a, c) \in R$;
   R is not an equivalence relation since R is not reflexive.

   (d) The relation R is reflexive since for any $a \in \mathbb{Z}$, $a \equiv a \pmod 3$, so $(a, a) \in R$;
   R is symmetric since if $(a, b) \in R$, then $a \equiv b \pmod 3$, then $b \equiv a \pmod 3$, and then $(b, a) \in R$;
   R is transitive since if $(a, b) \in R$ and $(b, c) \in R$, then $a \equiv b \pmod 3$ and $b \equiv c \pmod 3$, then $a \equiv c \pmod 3$, so $(a, c) \in R$;
   R is an equivalence relation since it is reflexive, symmetric, and transitive.
   The equivalence classes are $[0] = \{a \in \mathbb{Z} \mid a \equiv 0 \pmod 3\}$, $[1] = \{a \in \mathbb{Z} \mid a \equiv 1 \pmod 3\}$, and $[2] = \{a \in \mathbb{Z} \mid a \equiv 2 \pmod 3\}$.

   (e) The relation R is not reflexive: e.g. $(1, 1) \notin R$ since $1 \not> 1$;
   R is not symmetric: e.g. $(2, 1) \in R$ and $(1, 2) \notin R$;
   R is transitive since if $(a, b) \in R$ and $(b, c) \in R$, then $a > b$ and $b > c$, then

$a > c$, so $(a, c) \in R$;
R is not an equivalence relation since R is not symmetric.

4. (a) The function f is not one-to-one: e.g. $5 \cdot 1^2 + 2 = 5(-1)^2 = 2$, but $1 \neq -1$;
f is not onto: e.g. there is no integer n such that $5n + 2 = 3$ since the only real solution of this equation is $n = \frac{1}{5}$ which is not an integer;
f is not bijective: e.g. it is not one-to-one;

(b) The function f is one-to-one since if $\frac{1}{x} = \frac{1}{y}$, then $x = y$;
f is not onto: e.g. there is no natural number n such that $\frac{1}{n} = 2$ since the only real solution of this equation is $n = \frac{1}{2}$ which is not a natural number;
f is not bijective since it is not onto.

(c) The function f is one-to-one: let $f(x) = f(y)$ where $x, y \in \mathbb{R}$. If $f(x) \neq 0$, then $x \neq 0$ and $y \neq 0$, so $\frac{1}{x} = \frac{1}{y}$. Therefore $x = y$.
The function f is onto: let $y \in \mathbb{R}$. If $y \neq 0$, let $x = \frac{1}{y}$. Then $f(x) = \frac{1}{1/y} = y$. If $y = 0$, then $f(0) = y$.
This function is bijective since it is both one-to-one and onto.

(d) The function f is not one-to-one: e.g. $f(1) = f(0)$ but $1 \neq 0$;
f is onto since it is a continuous function with $\lim\limits_{x \to -\infty} = -\infty$ and $\lim\limits_{x \to \infty} = \infty$;
f is not bijective since it is not one-to-one.

5. Prove or disprove the following statements.

(a) The statement is false. Counterexample: $A = B = C = \{1, 2\}$, $f = \{(1, 1), (2, 1)\}$, $g = \{(1, 1), (2, 2)\}$, $g \circ f = \{(1, 1), (2, 1)\}$. Here g is onto, but $g \circ f$ is not.

(b) The statement is true. Let $f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$. Then $g(f(x_1)) = g(f(x_2))$. Since $g \circ f$ is one-to-one, $x_1 = x_2$. Thus f is one-to-one.
(Note: we did not use the fact that g is one-to-one.)

(c) The statement is false. Counterexample: $A = C = \{1\}$, $B = \{1, 2\}$, $f = \{(1, 1)\}$, $g = \{(1, 1), (2, 1)\}$, $g \circ f = \{(1, 1)\}$. Here both f and $g \circ f$ are one-to-one, but g is not.

6. (a) Proof by Mathematical Induction.
First we check the statement for $n = 1$: $1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$ is true.
Now suppose the statement holds for $n = k$ for some $k \in \mathbb{Z}$, i.e.
$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) = \dfrac{k(k+1)(k+2)}{3}$.
Adding $(k+1)(k+2)$ gives
$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) + (k+1)(k+2) = \dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \dfrac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} = \dfrac{(k+1)(k+2)(k+3)}{3}$.
Thus the statement holds for $n = k + 1$.

(b) Proof by Mathematical Induction.
First we check the statement for $n = 1$: using the Product rule and the Chain rule, we have $f'(x) = e^{-x} - xe^{-x} = (-1)e^{-x}(x-1)$ is true.
Now suppose the statement holds for $n = k$ for some $k \in \mathbb{Z}$, i.e.
$f^{(k)}(x) = (-1)^k e^{-x}(x-k)$. Differentiating both sides gives $f^{(k+1)}(x) = (-1)^k(-e^{-x}(x-k) + e^{-x}) = (-1)^{k+1}(e^{-x}(x-k) - e^{-x}) = (-1)^{k+1}e^{-x}(x-(k+1))$. Thus the statement holds for $n = k+1$.

(c) Proof by Mathematical Induction.
First we check the statement for $n = 1$: $5|(1^5 - 1)$ is true since $5|0$.
Now suppose the statement holds for $n = k$ for some $k \in \mathbb{Z}$, i.e.
$5|(k^5 - k)$. Then $k^5 - k = 5m$ for some $m \in \mathbb{Z}$. Therefore $(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 = (k^5 - k) + (5k^4 + 10k^3 + 10k^2 + 5k) = 5m + 5(k^4 + 2k^3 + 2k^2 + k) = 5(m + k^4 + 2k^3 + 2k^2 + k)$. Since $m + k^4 + 2k^3 + 2k^2 + k \in \mathbb{Z}$, $5|((k+1)^5 - (k+1))$. Thus the statement holds for $n = k+1$.