MATH 111
Test 1 - Solutions

1. Let $A = \{x \in \mathbb{N} \mid x \geq 4\}$ and $B = \{x \in \mathbb{N} \mid 3 \leq x \leq 6\}$, and let $\mathbb{N}$ be the universal set. Determine the following sets:

   (a) $A \cap B = \{x \in \mathbb{N} \mid 4 \leq x \leq 6\} = \{4, 5, 6\}$
   (b) $\overline{A} = \{x \in \mathbb{N} \mid x < 4\} = \{1, 2, 3\}$
   (c) $\overline{A} \cup B = \{x \in \mathbb{N} \mid x \leq 6\} = \{1, 2, 3, 4, 5, 6\}$

2. Let P and Q be propositions. Prove that the compound propositions $Q \Rightarrow \neg P$ and $\neg(P \wedge Q)$ are logically equivalent.

   We construct the truth table:

P	Q	$\neg P$	$Q \Rightarrow \neg P$	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	T	F	T

   We see that for any combination of truth values of P and Q, $Q \Rightarrow \neg P$ and $\neg(P \wedge Q)$ have the same truth values. Therefore these compound propositions are logically equivalent.

3. Let $C(x, y)$ denote "x and y are taking a class together" where x and y are students at Fresno State. Write in words the following statements and determine their truth values. Explain your reasons!

   (a) $\forall x \forall y C(x, y)$
   Every two Fresno State students are taking a class together. This statement is false, e.g. I and my friend Jon are both Fresno State students, but we are not taking a class together (or something like this; giving a specific example would be most convincing).

   (b) $\exists x \forall y C(x, y)$
   There is a student at Fresno State who is taking a class together with every Fresno State student. This statement is false, because I believe that no one is taking more than 20 classes, and there is no class that contains more than 1,000 students, so no one can take a class with more than 20,000 students. However, there are more than 20,000 students at Fresno State (or something like this; you probably won't be able to give a rigorous proof here, but you should explain your reason.)

   (c) $\forall x \exists y C(x, y)$
   Every Fresno State student is taking a class with some Fresno State student. Solution 1: If we assume that every class has at least two students or we allow $y = x$, then this statement is true. Solution 2: False, because some students are only taking one independent study class in which they are the only student, and saying that "x and x are taking a class together" would be grammatically incorrect. Therefore there is a student who is not taking a class with anybody else.

4. Let $S = \{1, 2\}$ and $T = \{2, 3, 4\}$. List the elements of $S \times T$.

   $(1, 2),\ (1, 3),\ (1, 4),\ (2, 2),\ (2, 3),\ (2, 4)$

5. Prove the following propositions. For each one, indicate what type of proof you are using.

   - Let $n \in \mathbb{Z}$. If $4n + 5$ is even, then $7n^2 - 3n + 8$ is odd.

     Since $4n+5 = 2(2n+2)+1$ and $2n+2 \in \mathbb{Z}$, $4n+5$ is odd. Therefore the implication is true.
     This is a vacuous proof.

   - Let $n \in \mathbb{Z}$. If n is odd, then $3n + 5$ is even.

     If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then $3n + 5 = 3(2k + 1) + 5 = 6k + 8 = 2(3k + 4)$. Since $3k + 4 \in \mathbb{Z}$, $3n + 5$ is even.
     This is a direct proof.

   - Let $n \in \mathbb{Z}$. If $3n + 5$ is even, then n is odd.

     We will prove the contrapositive proposition, i.e. that if n is even, then $3n + 5$ is odd.
     If n is even, then $n = 2k$ for some $k \in \mathbb{Z}$. Then $3n + 5 = 3(2k) + 5 = 6k + 5 = 2(3k + 2) + 1$. Since $3k + 3 \in \mathbb{Z}$, $3n + 5$ is odd.
     This is a proof by contrapositive.

6. (For extra credit) If $|A| = 21$, $|B| = 19$, $|C| = 17$, $|A \cap B| = 9$, $|A \cap C| = 8$, $|B \cap C| = 7$, $|(A \cap B) - C| = 6$, find $|A \cup B \cup C|$.

   Since $|A \cap B| = 9$ and $|(A \cap B) - C| = 6$, $|A \cap B \cap C| = 9 - 6 = 3$. Since $|A \cap C| = 8$, $|(A \cap C) - B| = 8 - 3 = 5$. Since $|B \cap C| = 7$, $|(B \cap C) - A| = 7 - 3 = 4$. Since $|A| = 21$, $|A - B - C| = 21 - 6 - 3 - 5 = 7$. Since $|B| = 19$, $|B - A - C| = 19 - 6 - 3 - 4 = 6$. Since $|C| = 17$, $|C - A - B| = 17 - 5 - 3 - 4 = 5$. Then $|A \cup B \cup C| = 7 + 6 + 6 + 5 + 3 + 4 + 5 = 36$ (see diagram below).