MATH 111
Test 3 (12/11/07) - Solutions

1. Let A and B be nonempty sets. Is it true or false that every function from A to B is also a relation from A to B? Explain.

   This is true. Every function is a relation: a function from A to B is defined as a relation from A to B such that every element of A appears as the first coordinate in exactly one pair in the relation (recall that a relation is a subset of $A \times B$).

2. Let A be a set and $f : A \to A$ be one-to-one. Prove that $f \circ f$ is one-to-one.

   Let $(f \circ f)(a) = (f \circ f)(b)$ for some $a, b \in A$. Then $f(f(a)) = f(f(b))$. Since f is one-to-one, if follows that $f(a) = f(b)$ and therefore $a = b$. Thus $f \circ f$ is one-to-one.

3. Determine whether $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2 + 3$ is one-to-one; onto; bijective.

   If $f(a) = f(b)$ for $a, b \in \mathbb{N}$, then $a^2 + 3 = b^2 + 3$. Then $a^2 = b^2$. Since $a, b > 0$, it follows that $a = b$. Thus f is one-to-one. Since the equation $x^2 + 3 = 1$, or equivalently, $x^2 = -2$, has no real solutions, $1 \in \mathbb{N}$ is not in the image of f. Thus f is not onto. Since f is not onto, it is not bijective.

4. Let R be a relation on $\mathbb{R}$ defined by $(a, b) \in R$ if and only if $a+b \in \mathbb{Z}$. Determine whether R is reflexive; symmetric; transitive; an equivalence relation. If it is an equivalence relation, describe its distinct equivalence classes.

   Since $0.1 + 0.1 = 0.2 \notin \mathbb{Z}$, $(0.1, 0.1) \notin R$. Thus R is not reflexive.

   If $(a, b) \in R$, then $a + b \in \mathbb{Z}$. Therefore $b + a = a + b \in \mathbb{Z}$. So $(b, a) \in R$. Thus R is symmetric.

   Since $0.1 + 0.9 = 1 \in \mathbb{Z}$ and $0.9 + 0.1 = 1 \in \mathbb{Z}$, but $0.1 + 0.1 = 0.2 \notin \mathbb{Z}$, we have $(0.1, 0.9) \in R$, $(0.9, 0.1) \in R$, but $(0.1, 0.1) \notin R$. Thus R is not transitive.

   Since R is not reflexive, it is not an equivalence relation.

5. Recall that the factorial of n is defined as $n! = 1 \cdot 2 \cdot 3 \cdot \ldots n$. Prove that for any positive integer n,

   $$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n + 1)! - 1.$$

   We will prove this identity by Mathematical Induction.

   Basis step: if $n = 1$, then $1 \cdot 1! = 2! - 1$ is true.

   Inductive step: assume that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k! = (k + 1)! - 1$ for some $k \in \mathbb{N}$. We will prove that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + (k + 1) \cdot (k + 1)! = (k + 2)! - 1$.

   Observe that

   $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + (k+1) \cdot (k+1)! = (1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots k \cdot k!) + (k+1) \cdot (k+1)! = (k+1)! - 1 + (k+1) \cdot (k+1)! = (k+1)!(1+k+1) - 1 = (k+1)!(k+2) - 1 = (k+2)! - 1.$

6. (For extra credit) Give an example of a bijective function from $\mathbb{Q}$ to $\mathbb{Q} - \{0\}$.

Consider $f : \mathbb{Q} \to (\mathbb{Q} - \{0\})$ *defined by*

$$f(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{N} \cup \{0\} \\ x & \text{otherwise} \end{cases}.$$

Note that $f(x) \in \mathbb{N}$ *if and only if* $x \in \mathbb{N} \cup \{0\}$.

Let $f(x_1) = f(x_2)$. *We will consider two cases.*

Case I: $f(x_1) \in \mathbb{N}$. *Then* $x_1, x_2 \in \mathbb{N} \cup \{0\}$, *so* $x_1 + 1 = x_2 + 1$. *Therefore* $x_1 = x_2$.

Case II: $f(x_1) \notin \mathbb{N}$. *Then* $x_1, x_2 \notin \mathbb{N} \cup \{0\}$, *so* $x_1 = f(x_1) = f(x_2) = x_2$.

Thus f *is one-to-one.*

Now let $y \in \mathbb{Q} - \{0\}$. *Again, we will consider two cases.*

Case I: $y \in \mathbb{N}$. *Let* $x = y - 1$. *Then* $x \in \mathbb{N} \cup \{0\}$, *and* $f(x) = x + 1 = y$.

Case II: $y \notin \mathbb{N} \cup \{0\}$, *let* $x = y$. *Then* $x \notin \mathbb{N} \cup \{0\}$, *and* $f(x) = x = y$.

Thus f *is onto.*

Since f *is one-to-one and onto, it is bijective.*