Math 111 May 8, 2006
Test 3 – Solutions

1. Let R be an equivalence relation defined on a set A containing the elements a, b, c, and d. Prove that if $a\ R\ b$, $c\ R\ d$, and $a\ R\ d$, then $b\ R\ c$.

   Since $a\ R\ b$ and R is symmetric, $b\ R\ a$. Since $b\ R\ a$, $a\ R\ d$, and R is transitive, $b\ R\ d$. Since $c\ R\ d$ and R is symmetric, $d\ R\ c$. Finally, since $b\ R\ d$, $d\ R\ c$, and R is transitive, we have $b\ R\ c$.

2. (a) Let $f : B \to C$ and $g : C \to D$ be functions such that $g \circ f$ is onto. Prove that g is onto.

   Let $d \in D$. Since $g \circ f$ is onto, there exists $b \in B$ such that $(g \circ f)(b) = d$. Then $g(f(b)) = d$. Let $c = f(b)$. Then $c \in C$ and $g(c) = d$. Therefore g is onto.

   (b) Give an example of the situation in part (a) in which f is not onto.

   Let $B = \{1, 2\}$, $C = \{3, 4, 5\}$, $D = \{6, 7\}$, $f = \{(1, 3), (2, 4)\}$, and $g = \{(3, 6), (4, 7), (5, 7)\}$. Then f is not onto because 5 is not in the image, but $g \circ f = \{(1, 6), (2, 7)\}$ is onto because every element of D is in the image.

3. Prove that $3 + 7 + 11 + \cdots + (4n - 1) = n(2n + 1)$ for all $n \geq 1$.

   We will prove by Mathematical Induction.

   Basis step: if $n = 1$, then $3 = 1(2 + 1)$ is true.

   Inductive step: assume that $3+7+11+\cdots+(4k-1) = k(2k+1)$ for some $k \in \mathbb{N}$. We will prove that $3+7+11+\cdots+(4(k+1)-1) = (k+1)(2(k+1)+1)$, i.e. $3+7+11+\cdots+(4k+3) = (k + 1)(2k + 3)$.

   Observe that $3 + 7 + 11 + \cdots + (4k + 3) = (3 + 7 + 11 + \cdots + (4k - 1)) + (4k + 3) = k(2k + 1) + (4k + 3) = 2k^2 + k + 4k + 3 = 2k^2 + 5k + 3 = (k + 1)(2k + 3)$.

4. Determine whether each of the following functions is one-to-one, onto, neither, or both. Prove your answers.

   (a) $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = \sqrt{x^2 + 7}$.

   We will prove that the function f is neither one-to-one nor onto. It is not one-to-one because e.g. $f(1) = \sqrt{8} = f(-1)$, but $1 \neq -1$. It is not onto because e.g. 0 is not in the image as the equation $\sqrt{x^2 + 7} = 0$ has no real solutions (the only solutions are those of $x^2 = -7$, i.e. $x = \pm\sqrt{7}i$, but these are not real numbers).

   (b) $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{1\}$, given by $f(x) = \frac{x}{x-3}$.

   We will prove that the function f is both one-to-one and onto. Let $f(x_1) = f(x_2)$, then $\frac{x_1}{x_1-3} = \frac{x_2}{x_2-3}$. Then $x_1(x_2 - 3) = x_2(x_1 - 3)$, i.e. $x_1x_2 - 3x_1 = x_2x_1 - 3x_2$. It follows that $-3x_1 = -3x_2$, therefore $x_1 = x_2$. Thus f is one-to-one. Next, for any $y \in \mathbb{R} - \{1\}$, let $x = \frac{3y}{y-1}$. Then $x \in \mathbb{R} - \{3\}$ (since the equation $\frac{3y}{y-1} = 3$ has no solutions) and $f(x) = f\left(\frac{3y}{y-1}\right) = \frac{\frac{3y}{y-1}}{\frac{3y}{y-1} - 3} = \frac{3y}{3y - 3(y - 1)} = \frac{3y}{3} = y$. Thus f is onto.

5. A relation R is defined on $\mathbb{Z}$ by $a\ R\ b$ if $5a \equiv 2b \pmod 3$. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

   (1) Since $5 \equiv 2\ (mod\ 3)$, it follows that for any $a \in \mathbb{Z}$, $5a \equiv 2a\ (mod\ 3)$. So $a\ R\ a$. Thus R is reflexive.

   (2) If $a\ R\ b$, then $5a \equiv 2b\ (mod\ 3)$. Since $5 \equiv 2\ (mod\ 3)$, it follows that $5b \equiv 2b \equiv 5a \equiv 2a\ (mod\ 3)$. So $b\ R\ a$. Thus R is symmetric.

   (3) If $a\ R\ b$ and $b\ R\ c$, then $5a \equiv 2b\ (mod\ 3)$ and $5b \equiv 2c\ (mod\ 3)$. Then $5a \equiv 2b \equiv 5b \equiv 2c\ (mod\ 3)$. So $a\ R\ c$. Thus R is transitive.

   Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

   The equivalence classes are:

   $[0] = \{a \in \mathbb{Z} \mid 5a \equiv 0\ (mod\ 3)\} = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$,

   $[1] = \{a \in \mathbb{Z} \mid 5a \equiv 2\ (mod\ 3)\} = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$,

   $[2] = \{a \in \mathbb{Z} \mid 5a \equiv 4\ (mod\ 3)\} = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$;

   since $[0] \cup [1] \cup [2] = \mathbb{Z}$, these are all equivalence classes.

6. Prove that $24 \mid (5^{2n} - 1)$ for every positive integer n.

   We will prove by Mathematical Induction.

   Basis step: if $n = 1$, then $24|(5^2 - 1)$ is true.

   Inductive step: assume that $24|(5^{2k} - 1)$ for some $k \in \mathbb{N}$. We will prove that $24|(5^{2k+2} - 1)$.

   Since $24|(5^{2k} - 1)$, $5^{2k} - 1 \equiv 0\ (mod\ 24)$. Then $5^{2k+2} - 1 \equiv 5^{2k} \cdot 5^2 - 1 \equiv 5^{2k} \cdot 25 - 1 \equiv 5^{2k} \cdot 1 - 1 \equiv 5^{2k} - 1 \equiv 0\ (mod\ 24)$. Thus $24|(5^{2k+2} - 1)$.

Extra Credit

If $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, prove that $\alpha^{3n} = \underbrace{\alpha \circ \alpha \circ \cdots \circ \alpha}_{3n} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ for all $n \geq 1$.

Since $\alpha^3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$,

$\alpha^{3n} = (\alpha^3)^n = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$