MATH 111
Test 2 - Solutions

1. Let $x \in \mathbb{Z}$. Prove that if $x^3 + x^2 + x$ is odd, then x is odd. What type of proof (direct, by contrapositive, or by contradiction) did you use?

   We will prove this statement by contrapositive. If x is even, then $x = 2k$ for some $k \in \mathbb{Z}$. Then $x^3 + x^2 + x = (2k)^3 + (2k)^2 + 2k = 8k^3 + 4k^2 + 2k = 2(4k^3 + 2k^2 + k)$. Since $4k^3 + 2k^2 + k \in \mathbb{Z}$, $x^3 + x^2 + x$ is even.

2. Prove or disprove.

   The sum of two irrational numbers is irrational.

   The statement is false. Counterexample: $\sqrt{2} + (-\sqrt{2}) = 0$. We know that $\sqrt{2}$ is irrational. Below we show that $-\sqrt{2}$ is also irrational. However, $0 = \frac{0}{1}$ is rational.

   We will show that $-\sqrt{2}$ is irrational by contradiction. Suppose it is rational, then $-\sqrt{2} = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n \neq 0$. Then $\sqrt{2} = -\frac{m}{n} = \frac{-m}{n}$. Since $m, n \in \mathbb{Z}$ and $n \neq 0$, $\sqrt{2}$ is rational. However, this is not the case. Therefore $-\sqrt{2}$ is irrational.

3. Prove or disprove.

   The sum of a rational number and an irrational number is irrational.

   The statement is true. We will prove it by contradiction, namely, suppose there exist a rational number r and an irrational number x such that $r + x$ is rational. Then $r = \frac{k}{l}$ and $r + x = \frac{m}{n}$ for some $k, l, m, n \in \mathbb{Z}$, $l \neq 0$, $n \neq 0$. Then $x = (r + x) - r = \frac{m}{n} - \frac{k}{l} = \frac{ml - nk}{nl}$. Since $ml - nk \in \mathbb{Z}$ and $nl \neq 0$, x is rational. We get a contradiction.

4. Prove or disprove.

   Let A, B, and C be sets. If $A \cup B = A \cup C$, then $B = C$.

   The statement is false. Counterexample: $A = \{1, 2, 3\}$, $B = \{1\}$, $C = \{2\}$. Then $A \cup B = \{1, 2, 3\} = A \cup C$, but $B \neq C$.

5. Let $A = \{1, 2, 3, 4\}$. Give an example of a relation on A that is symmetric but not reflexive.

   $R = \{(1, 2), (2, 1)\}$ is symmetric (because for any $(a, b) \in R$, $(b, a) \in R$), but not reflexive (because e.g. $(1, 1) \notin R$).

6. Let R be the relation on $\mathbb{Z}$ defined by $(a, b) \in R$ iff $a + b < 5$.

   (a) Is R reflexive?
   No, because e.g. $(3, 3) \notin R$ since $3 + 3 \not< 5$.

(b) Is R symmetric?

Yes, because if $(a,b) \in R$, *then* $a+b<5$, *then* $b+a<5$, *so* $(b,a) \in R$.

(c) Is R transitive?

No, because e.g. $(2,1) \in R$ *and* $(1,3) \in R$, *but* $(2,3) \notin R$ *since* $2+1<5$, $1+3<5$, *but* $2+3 \not< 5$.

7. Prove or disprove.

The number $\sqrt{3}+\sqrt{5}$ is irrational.

The statement is true. We will prove it by contradiction. Suppose $x=\sqrt{3}+\sqrt{5}$ *is rational. Then* $x=\dfrac{m}{n}$ *for some* $m,n \in \mathbb{Z}$, $n \neq 0$. *Since* $x>0$, $m \neq 0$. *We have* $x-\sqrt{3}=\sqrt{5}$. *Then* $(x-\sqrt{3})^2=(\sqrt{5})^2$, *so* $x^2-2x\sqrt{3}+3=5$. *This implies that* $\sqrt{3}=\dfrac{x^2-2}{2x}=\dfrac{\frac{m^2}{n^2}-2}{2\frac{m}{n}}=\dfrac{\frac{m^2-2n^2}{n^2}}{\frac{2m}{n}}=\dfrac{m^2-2n^2}{2mn}$. *Since* $m^2-2n^2, 2mn \in \mathbb{Z}$ *and* $2mn \neq 0$, $\sqrt{3}$ *is rational. However, by a homework problem* $\sqrt{3}$ *is irrational. Contradiction. Therefore* x *is irrational.*