Homework 5 - Solutions

Q.1. (a) \( \exists x \neg P(x) \) is true. For example, if \( x = -1 \), then \( P(x) \) is false, so \( \neg P(x) \) is true.

(b) \( \forall x (P(x) \lor Q(x)) \) is false. For example, if \( x = 0 \), then both \( P(x) \) and \( Q(x) \) are false.

(c) \( \exists x (P(x) \land Q(x)) \) is true. For example, if \( x = 1 \), then both \( P(x) \) and \( Q(x) \) are true.

(d) \( \forall x (P(x) \Rightarrow Q(x)) \) is true because the square of any positive number is positive.

(e) \( \exists x (Q(x) \Rightarrow P(x)) \) is true. For example, if \( x = 1 \), then both \( P(x) \) and \( Q(x) \) are true, so the implication \( Q(x) \Rightarrow P(x) \) is true.

(f) \( \forall x (P(x) \Leftrightarrow Q(x)) \) is false. For example, if \( x = -1 \), then \( P(x) \) is false and \( Q(x) \) is true, so the biconditional \( P(x) \Leftrightarrow Q(x) \) is false.

Q.2. Are propositions

(a) Propositions \( \forall x (P(x) \Leftrightarrow Q(x)) \) and \( (\forall x P(x)) \Leftrightarrow (\forall x Q(x)) \) are not logically equivalent.

Let \( P(x) \) denote “\( x > 0 \)” and let \( Q(x) \) denote “\( x < 0 \)” where \( x \in \mathbb{R} \). Then \( \forall x (P(x) \Leftrightarrow Q(x)) \) is false: for example, if \( x = 1 \), then \( P(x) \) is true and \( Q(x) \) is false. However, \( (\forall x P(x)) \Leftrightarrow (\forall x Q(x)) \) is true because both \( \forall x P(x) \) and \( \forall x Q(x) \) are false (e.g. \( P(x) \) is false for \( x = -1 \) and \( Q(x) \) is false for \( x = 1 \)).

(b) Propositions \( \exists x (P(x) \Leftrightarrow Q(x)) \) and \( (\exists x P(x)) \Leftrightarrow (\exists x Q(x)) \) are not logically equivalent.

Let \( P(x) \) denote “\( x < 0 \)” and let \( Q(x) \) denote “\( x^2 < 0 \)”.
Then \( \exists x (P(x) \Leftrightarrow Q(x)) \) is true: for example, if \( x = 1 \), then both \( P(x) \) and \( Q(x) \) are false. However, \( (\exists x P(x)) \Leftrightarrow (\exists x Q(x)) \) is false because \( \exists x P(x) \) is true (e.g. for \( x = -1 \)) and \( \exists x Q(x) \) is false (since the square of any real number is nonnegative).

4.2. If \( a \mid b \) and \( b \mid a \), then by definition \( b = ac \) for some \( c \in \mathbb{Z} \) and \( a = bd \) for some \( d \in \mathbb{Z} \). Then \( a = bd = acd \), therefore \( a(cd - 1) = 0 \). Since \( a \neq 0 \), it follows that \( cd - 1 = 0 \), so \( cd = 1 \). The only pairs of integers whose product is 1 are \( 1 \cdot 1 = 1 \) and \( (-1) \cdot (-1) = 1 \). If \( c = d = 1 \), then \( a = b \). If \( c = d = -1 \), then \( a = -b \).

4.4. If \( 3 \nmid x \), then either \( x = 3k + 1 \) or \( x = 3k + 2 \) for some \( k \in \mathbb{Z} \). If \( 3 \nmid y \), then either \( y = 3l + 1 \) or \( y = 3l + 2 \) for some \( l \in \mathbb{Z} \). Thus we have four cases.

Case 1: \( x = 3k + 1 \) and \( y = 3l + 1 \). Then \( x^2 - y^2 = (3k + 1)^2 - (3l + 1)^2 = 9k^2 + 6k + 1 - 9l^2 - 6l - 1 = 9k^2 + 6k - 9l^2 - 6l = 3(3k^2 + 2k - 3l^2 - 2l) \). Since \( 3k^2 + 2k - 3l^2 - 2l \in \mathbb{Z} \), \( x^2 - y^2 \) is divisible by 3.
Case II: $x = 3k + 1$ and $y = 3l + 2$. Then $x^2 - y^2 = (3k + 1)^2 - (3l + 2)^2 = 9k^2 + 6k + 1 - 9l^2 - 12l - 4 = 9k^2 + 6k - 9l^2 - 12l - 3 = 3(3k^2 + 2k - 3l^2 - 4l - 1)$. Since $3k^2 + 2k - 3l^2 - 4l - 1 \in \mathbb{Z}$, $x^2 - y^2$ is divisible by 3.

Case III: $x = 3k + 2$ and $y = 3l + 2$. Then $x^2 - y^2 = (3k + 2)^2 - (3l + 2)^2 = 9k^2 + 12k + 4 - 9l^2 - 12l - 4 = 9k^2 + 12k - 9l^2 - 12l + 3 = 3(3k^2 + 4k - 3l^2 - 2l + 1)$. Since $3k^2 + 4k - 3l^2 - 2l + 1 \in \mathbb{Z}$, $x^2 - y^2$ is divisible by 3.

Case IV: $x = 3k + 2$ and $y = 3l + 2$. Then $x^2 - y^2 = (3k + 2)^2 - (3l + 2)^2 = 9k^2 + 12k + 4 - 9l^2 - 12l - 4 = 9k^2 + 12k - 9l^2 - 12l + 3 = 3(3k^2 + 4k - 3l^2 - 4l)$. Since $3k^2 + 4k - 3l^2 - 4l \in \mathbb{Z}$, $x^2 - y^2$ is divisible by 3.

4.6. We will prove this by contrapositive. If $3 \nmid a$, then either $a = 3k + 1$ or $a = 3k + 2$ for some $k \in \mathbb{Z}$.

Case I: $a = 3k + 1$ for some $k \in \mathbb{Z}$. Then $2a = 3k + 2$, therefore $3 \nmid 2a$.

Case II: $a = 3k + 2$ for some $k \in \mathbb{Z}$. Then $2a = 3k + 4 = 3(k + 1) + 1$, therefore $3 \nmid 2a$.

4.8. If $a \equiv b \pmod{n}$, then $n|(a - b)$, i.e. $a - b = nk$ for some $k \in \mathbb{Z}$. Then $a^2 - b^2 = (a - b)(a + b) = (nk)(a + b) = n(k(a + b))$, therefore $n|(a^2 - b^2)$, so $a^2 \equiv b^2 \pmod{n}$.

4.12. Let $n \in \mathbb{Z}$. If $n \not\equiv 0 \pmod{3}$ and $n \not\equiv 1 \pmod{3}$, then $n^2 \not\equiv n \pmod{3}$.

Since the conjunction $n \not\equiv 0 \pmod{3}$ and $n \not\equiv 1 \pmod{3}$ is equivalent to $n \equiv 2 \pmod{3}$, the above can also be written as follows: if $n \equiv 2 \pmod{3}$, then $n^2 \not\equiv n \pmod{3}$.

Proof. If $n \equiv 2 \pmod{3}$, then $n = 3k + 2$ for some $k \in \mathbb{Z}$. Then $n^2 = (3k + 2)^2 = 9k^2 + 12k + 4$, and $n^2 - n = 9k^2 + 12k - 9k - 4 - (3k + 2) = 9k^2 + 9k + 2 = 3(3k^2 + 3k) + 2$. Therefore $3 \nmid n^2 - n$, so $n^2 \not\equiv n \pmod{3}$.

Let $n \in \mathbb{Z}$. Then $n^2 \not\equiv n \pmod{3}$ if and only if $n \not\equiv 0 \pmod{3}$ and $n \not\equiv 1 \pmod{3}$.

This can be written as follows: let $n \in \mathbb{Z}$. Then $n^2 \not\equiv n \pmod{3}$ if and only if $n \equiv 2 \pmod{3}$.