MATH 111 **Spring 2006**

Practice Test 2 - Solutions

1. Read the textbook!
2. (a) If n is an integer such that $5|(n-1)$, then $n \equiv 1 \pmod 5$. Then $n^3+n-2 \equiv 1^3+1-2 \equiv 0 \pmod 5$. This implies that $5|(n^3+n-2)$. (This is a direct proof.)
Another proof: If n is an integer such that $5|(n-1)$, then $n-1=5k$ for some $k \in \mathbb{Z}$. Then $n=5k+1$, therefore $n^3+n-2=(5k+1)^3+(5k+1)-2=125k^3+75k^2+15k+1+5k+1-2=125k^3+75k^2+20k=5(25k^3+15k^2+4k)$. Since $25k^3+15k^2+4k \in \mathbb{Z}$, $5|(n^3+n-2)$. (This is also a direct proof.)

   (b) Assume that $\log_3 2$ is rational. Then $\log_3 2 = \dfrac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n > 0$. Then $3^{\frac{m}{n}} = 2$, so $3^m = 2^n$. Since $n > 0$, $3^m = 2^n > 1$, so $m > 0$. Since $3 \equiv 1 \pmod 2$, $3^m \equiv 1 \pmod 2$, so 3^m is odd. However, $2^n = 2 \cdot 2^{n-1}$ is even. We get a contradiction. Therefore $\log_3 2$ is irrational. (This is a proof by contradiction.)

   (c) We will prove this statement by contrapositive. Assume that n is odd. Then $n=2k+1$ for some $k \in \mathbb{Z}$. Then $7n^2+4=7(2k+1)^2+4=7(4k^2+4k+1)+4=28k^2+28k+11=2(14k^2+14k+5)+1$. Since $14k^2+14k+5 \in \mathbb{Z}$, $7n^2+4$ is odd.

   (d) First we will prove that if $3|(mn)$ then $3|m$ or $3|n$. We will prove this by contrapositive, namely, we will prove that if $3 \not| m$ and $3 \not| n$, then $3 \not| (mn)$. If $3 \not| m$, then $m=3k+1$ or $m=3k+2$ for some $k \in \mathbb{Z}$. If $3 \not| n$, then $n=3l+1$ or $n=3l+2$ for some $l \in \mathbb{Z}$. Thus we have four cases:
<u>Case I:</u> $m=3k+1$, $n=3l+1$. Then $mn=(3k+1)(3l+1)=9kl+3k+3l+1=3(3kl+k+l)+1$. Since $3kl+k+l \in \mathbb{Z}$, $3 \not| (mn)$.
<u>Case II:</u> $m=3k+1$, $n=3l+2$. Then $mn=(3k+1)(3l+2)=9kl+6k+3l+2=3(3kl+2k+l)+2$. Since $3kl+2k+l \in \mathbb{Z}$, $3 \not| (mn)$.
<u>Case III:</u> $m=3k+2$, $n=3l+1$. Then $mn=(3k+2)(3l+1)=9kl+3k+6l+2=3(3kl+k+2l)+2$. Since $3kl+k+2l \in \mathbb{Z}$, $3 \not| (mn)$.
<u>Case IV:</u> $m=3k+2$, $n=3l+2$. Then $mn=(3k+2)(3l+2)=9kl+6k+6l+4=3(3kl+2k+2l+1)+1$. Since $3kl+2k+2l+1 \in \mathbb{Z}$, $3 \not| (mn)$.
Next we will prove that if $3|m$ or $3|n$, then $3|(mn)$. Here we have two cases:
<u>Case I:</u> $3|m$. Then $m=3k$ for some $k \in \mathbb{Z}$. Then $mn=3kn$. Since $kn \in \mathbb{Z}$, $3|(mn)$.
<u>Case II:</u> $3|n$. Then $n=3l$ for some $l \in \mathbb{Z}$. Then $mn=m3l=3ml$. Since $ml \in \mathbb{Z}$, $3|(mn)$.
(This direction we proved directly.)

   (e) Assume that there exist a nonzero rational number x and an irrational number

y such that xy is rational. Then $x = \dfrac{k}{l}$ for some $k, l \in \mathbb{Z}$, $k \neq 0$ and $l \neq 0$, and $xy = \dfrac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n \neq 0$. Then $y = \dfrac{xy}{x} = \dfrac{\frac{m}{n}}{\frac{k}{l}} = \dfrac{ml}{nk}$. Since $ml, nk \in \mathbb{Z}$ and $nk \neq 0$, y is rational. Contradiction. (This is a proof by contradiction.)

(f) We will prove this statement by contrapositive. Namely, we will assume that $a|b$ or $a|c$ and we will show that $a|(bc)$. If $a|b$, then $b = ak$ for some $k \in \mathbb{Z}$, and $bc = akc$. Since $kc \in \mathbb{Z}$, $a|(bc)$. If $a|c$, then $c = ak$ for some $k \in \mathbb{Z}$, and $bc = bak = abk$. Since $bk \in \mathbb{Z}$, $a|(bc)$.

3. (a) This statement is true. For example, if $a = -1$, then for every real number b, we have $b^2 \geq 0 \geq -1$, so $b^2 \geq a$.

   (b) This statement is false. For any integer a, either $a \leq 4$ or $a \geq 5$. If $a \leq 4$, then $a^3 + 2a + 3 \leq 64 + 8 + 3 = 75 < 100$, so $a^3 + 2a + 3 \neq 100$. If $a \geq 5$, then $a^3 + 2a + 3 \geq 125 + 10 + 3 = 138 > 100$, so $a^3 + 2a + 3 \neq 100$.

   (c) This statement is true. For any sets A and B, let $C = A \cup B$. Then $A \cup C = A \cup A \cup B = A \cup B$ and $B \cup C = B \cup A \cup B = A \cup B$, so $A \cup C = B \cup C$.

   (d) This statement is false. For example, if $A = \{1\}$, $B = \{2\}$, $C = \{1, 2\}$, $D = \{2, 3\}$, then $A \subset C$, $B \subset D$, and $A \cap B = \emptyset$, however, $C \cap D \neq \emptyset$.

   (e) This statement if true. Suppose that $A \subset C$, $B \subset D$, $C \cap D = \emptyset$, but $A \cap B \neq \emptyset$. Then there is an element $x \in A \cap B$, so $x \in A$ and $x \in B$. Since $A \subset C$ and $B \subset D$, it follows that $x \in C$ and $x \in D$. Then $x \in C \cap D$, thus $C \cap D \neq \emptyset$. We get a contradiction.

4. (a) This set is not a relation from A to B because it is not a subset of $A \times B$: e.g. $(a, 1) \notin A \times B$.

   (b) This set is a relation from A to B since it is a subset of $A \times B$ (it is easy to see that each element of this set is of required form).

5. Determine which of the following relations are reflexive; symmetric; transitive.

   (a) R is not reflexive because e.g. $(1, 1) \notin R$ since $1 + 1 \neq 0$.
   R is symmetric because if $(a, b) \in R$, then $a + b = 0$, then $b + a = 0$, so $(b, a) \in R$.
   R is not transitive because e.g. $(1, -1) \in R$ and $(-1, 1) \in R$, however, $(1, 1) \notin R$.

   (b) R is reflexive because for any $a \in \mathbb{R}$, $\dfrac{a}{a} = 1 \in \mathbb{Q}$, so $(a, a) \in R$.
   R is not symmetric because e.g. $(0, 1) \in R$ since $\dfrac{0}{1} \in \mathbb{Q}$, but $(1, 0) \notin R$ since $\dfrac{1}{0}$ is undefined (and thus is not an element of $\mathbb{Q}$).
   R is transitive because if $(a, b) \in R$ and $(b, c) \in R$, then $\dfrac{a}{b} \in \mathbb{Q}$ and $\dfrac{b}{c} \in \mathbb{Q}$.

Since the product of two rational numbers is rational (see proof below), $\frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c} \in \mathbb{Q}$, thus $(a, c) \in R$.
Proof that the product of two rational numbers is rational: let $x, y \in \mathbb{Q}$, then $x = \frac{k}{l}$ and $y = \frac{m}{n}$ for some $k, l, m, n \in \mathbb{Z}$, $l \neq 0$, $n \neq 0$. Then $xy = \frac{k}{l} \cdot \frac{m}{n} = \frac{km}{ln}$. Since $km, ln \in \mathbb{Z}$ and $ln \neq 0$, $xy \in \mathbb{Q}$.

(c) R is not reflexive because $(0, 0) \notin R$ since $0 \cdot 0 \not> 0$.
R is symmetric because if $(a, b) \in R$, then $ab > 0$, then $ba > 0$, so $(b, a) \in R$.
R is transitive because if $(a, b) \in R$ and $(b, c) \in R$, then $ab > 0$ and $bc > 0$. Therefore $acb^2 > 0$. We know that $b^2 \geq 0$ for all $b \in \mathbb{R}$. Since $acb^2 \neq 0$, $b^2 \neq 0$. Therefore $b^2 > 0$, thus $ac > 0$.

(d) R is reflexive since for any $a \in \mathbb{Z}$, $a \equiv a \pmod 3$, thus $(a, a) \in R$.
R is symmetric because if $(a, b) \in R$, then $a \equiv b \pmod 3$, then $b \equiv a \pmod 3$, thus $(b, a) \in R$.
R is transitive because if $(a, b) \in R$ and $(b, c) \in R$, then $a \equiv b \pmod 3$ and $b \equiv c \pmod 3$, therefore $a \equiv c \pmod 3$, thus $(a, c) \in R$.

(e) R is not reflexive because e.g. $(1, 1) \notin R$ since $1 \not> 1$.
R is not symmetric because e.g. $(2, 1) \in R$ but $(1, 2) \notin R$ since $2 > 1$ but $1 \not> 2$.
R is transitive because if if $(a, b) \in R$ and $(b, c) \in R$, then $a > b$ and $b > c$, then $a > c$, thus $(a, c) \in R$.