1. Read the textbook!

2. (a) If $n$ is an integer such that $5|(n-1)$, then $n \equiv 1 \pmod{5}$. Then $n^3 + n - 2 \equiv 1^3 + 1 - 2 \equiv 0 \pmod{5}$. This implies that $5|(n^3 + n - 2)$. (This is a direct proof.)

Another proof: If $n$ is an integer such that $5|(n-1)$, then $n-1 = 5k$ for some $k \in \mathbb{Z}$. Then $n = 5k + 1$, therefore $n^3 + n - 2 = (5k + 1)^3 + (5k + 1) - 2 = 125k^3 + 75k^2 + 15k + 1 + 5k + 1 - 2 = 125k^3 + 75k^2 + 20k = 5(25k^3 + 15k^2 + 4k)$. Since $25k^3 + 15k^2 + 4k \in \mathbb{Z}$, $5|(n^3 + n - 2)$. (This is also a direct proof.)

(b) Assume that $\log_3 2$ is rational. Then $\log_3 2 = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n > 0$. Then $3^{\frac{m}{n}} = 2$, so $3^m = 2^n$. Since $n > 0$, $3^m = 2^n > 1$, so $m > 0$. Since $3 \equiv 1 \pmod{2}$, $3^m \equiv 1 \pmod{2}$, so $3^m$ is odd. However, $2^n = 2 \cdot 2^{n-1}$ is even. We get a contradiction. Therefore $\log_3 2$ is irrational. (This is a proof by contradiction.)

(c) We will prove this statement by contrapositive. Assume that $n$ is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then $7n^2 + 4 = 7(2k+1)^2 + 4 = 7(4k^2 + 4k + 1) + 4 = 28k^2 + 28k + 11 = 2(14k^2 + 14k + 5) + 1$. Since $14k^2 + 14k + 5 \in \mathbb{Z}$, $7n^2 + 4$ is odd.

(d) First we will prove that if $3|(mn)$ then $3|m$ or $3|n$. We will prove this by contrapositive, namely, we will prove that if $3 \nmid m$ and $3 \nmid n$, then $3 \nmid (mn)$.

If $3 \nmid m$, then $m = 3k + 1$ or $m = 3k + 2$ for some $k \in \mathbb{Z}$. If $3 \nmid n$, then $n = 3l + 1$ or $n = 3l + 2$ for some $l \in \mathbb{Z}$. Thus we have four cases:

- **Case I:** $m = 3k + 1$, $n = 3l + 1$. Then $mn = (3k+1)(3l+1) = 9kl + 3k + 3l + 1 = 3(3kl + k + l) + 1$. Since $3kl + k + l \in \mathbb{Z}$, $3 \nmid (mn)$.
- **Case II:** $m = 3k + 1$, $n = 3l + 2$. Then $mn = (3k+1)(3l+2) = 9kl + 6k + 3l + 2 = 3(3kl + 2k + l) + 2$. Since $3kl + 2k + l \in \mathbb{Z}$, $3 \nmid (mn)$.
- **Case III:** $m = 3k + 2$, $n = 3l + 1$. Then $mn = (3k+2)(3l+1) = 9kl + 3k + 6l + 2 = 3(3kl + k + 2l) + 2$. Since $3kl + k + 2l \in \mathbb{Z}$, $3 \nmid (mn)$.
- **Case IV:** $m = 3k + 2$, $n = 3l + 2$. Then $mn = (3k+2)(3l+2) = 9kl + 6k + 6l + 4 = 3(3kl + 2k + 2l + 1) + 1$. Since $3kl + 2k + 2l + 1 \in \mathbb{Z}$, $3 \nmid (mn)$.

Next we will prove that if $3|m$ or $3|n$, then $3|(mn)$. Here we have two cases:

- **Case I:** $3|m$. Then $m = 3k$ for some $k \in \mathbb{Z}$. Then $mn = 3kn$. Since $kn \in \mathbb{Z}$, $3|(mn)$.
- **Case II:** $3|n$. Then $n = 3l$ for some $l \in \mathbb{Z}$. Then $mn = m3l = 3ml$. Since $ml \in \mathbb{Z}$, $3|(mn)$.

(This direction we proved directly.)

(e) Assume that there exist a nonzero rational number $x$ and an irrational number
y such that $xy$ is rational. Then $x = \frac{k}{l}$ for some $k, l \in \mathbb{Z}$, $k \neq 0$ and $l \neq 0$, and $xy = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n \neq 0$. Then $y = \frac{xy}{x} = \frac{m}{l} = \frac{ml}{nk}$. Since $ml, nk \in \mathbb{Z}$ and $nk \neq 0$, $y$ is rational. Contradiction. (This is a proof by contradiction.)

(f) We will prove this statement by contrapositive. Namely, we will assume that $a|b$ or $a|c$ and we will show that $a|(bc)$. If $a|b$, then $b = ak$ for some $k \in \mathbb{Z}$, and $bc = akc$. Since $kc \in \mathbb{Z}$, $a|(bc)$. If $a|c$, then $c = ak$ for some $k \in \mathbb{Z}$, and $bc = bak = abk$. Since $bk \in \mathbb{Z}$, $a|(bc)$.

3. (a) This statement is true. For example, if $a = -1$, then for every real number $b$, we have $b^2 \geq 0 \geq -1$, so $b^2 \geq a$.

(b) This statement is false. For any integer $a$, either $a \leq 4$ or $a \geq 5$. If $a \leq 4$, then $a^3 + 2a + 3 \leq 64 + 8 + 3 = 75 < 100$, so $a^3 + 2a + 3 \neq 100$. If $a \geq 5$, then $a^3 + 2a + 3 \geq 125 + 10 + 3 = 138 > 100$, so $a^3 + 2a + 3 \neq 100$.

(c) This statement is true. For any sets $A$ and $B$, let $C = A \cup B$. Then $A \cup C = A \cup A \cup B = A \cup B$ and $B \cup C = B \cup A \cup B = A \cup B$, so $A \cup C = B \cup C$.

(d) This statement is false. For example, if $A = \{1\}$, $B = \{2\}$, $C = \{1, 2\}$, $D = \{2, 3\}$, then $A \subset C$, $B \subset D$, and $A \cap B = \emptyset$, however, $C \cap D \neq \emptyset$.

(e) This statement is true. Suppose that $A \subset C$, $B \subset D$, $C \cap D = \emptyset$, but $A \cap B \neq \emptyset$. Then there is an element $x \in A \cap B$, so $x \in A$ and $x \in B$. Since $A \subset C$ and $B \subset D$, it follows that $x \in C$ and $x \in D$. Then $x \in C \cap D$, thus $C \cap D \neq \emptyset$. We get a contradiction.

4. (a) This set is not a relation from $A$ to $B$ because it is not a subset of $A \times B$: e.g. $(a, 1) \not\in A \times B$.

(b) This set is a relation from $A$ to $B$ since it is a subset of $A \times B$ (it is easy to see that each element of this set is of required form).

5. Determine which of the following relations are reflexive; symmetric; transitive.

(a) $R$ is not reflexive because e.g. $(1, 1) \not\in R$ since $1 + 1 \neq 0$.

$R$ is symmetric because if $(a, b) \in R$, then $a + b = 0$, then $b + a = 0$, so $(b, a) \in R$.

$R$ is not transitive because e.g. $(1, -1) \in R$ and $(-1, 1) \in R$, however, $(1, 1) \not\in R$.

(b) $R$ is reflexive because for any $a \in \mathbb{R}$, $\frac{a}{a} = 1 \in \mathbb{Q}$, so $(a, a) \in R$.

$R$ is not symmetric because e.g. $(0, 1) \in R$ since $\frac{0}{1} \in \mathbb{Q}$, but $(1, 0) \not\in R$ since $\frac{1}{0}$ is undefined (and thus is not an element of $\mathbb{Q}$).

$R$ is transitive because if $(a, b) \in R$ and $(b, c) \in R$, then $\frac{a}{b} \in \mathbb{Q}$ and $\frac{b}{c} \in \mathbb{Q}$. 

Since the product of two rational numbers is rational (see proof below), \( \frac{a}{b} \cdot \frac{c}{d} \in \mathbb{Q} \), thus \((a, c) \in R\).

Proof that the product of two rational numbers is rational: let \(x, y \in \mathbb{Q}\), then \(x = \frac{k}{l}\) and \(y = \frac{m}{n}\) for some \(k, l, m, n \in \mathbb{Z}\), \(l \neq 0\), \(n \neq 0\). Then \(xy = \frac{k}{l} \cdot \frac{m}{n} = \frac{km}{ln}\). Since \(km, ln \in \mathbb{Z}\) and \(ln \neq 0\), \(xy \in \mathbb{Q}\).

(c) \(R\) is not reflexive because \((0, 0) \notin R\) since \(0 \cdot 0 \neq 0\).

R is symmetric because if \((a, b) \in R\), then \(ab > 0\), then \(ba > 0\), so \((b, a) \in R\).

(c) \(R\) is transitive because if \((a, b) \in R\) and \((b, c) \in R\), then \(ab > 0\) and \(bc > 0\). Therefore \(acb^2 > 0\). We know that \(b^2 \geq 0\) for all \(b \in \mathbb{R}\). Since \(acb^2 \neq 0\), \(b^2 \neq 0\). Therefore \(b^2 > 0\), thus \(ac > 0\).

(d) \(R\) is reflexive since for any \(a \in \mathbb{Z}\), \(a \equiv a \pmod{3}\), thus \((a, a) \in R\).

R is symmetric because if \((a, b) \in R\), then \(a \equiv b \pmod{3}\), then \(b \equiv a \pmod{3}\), thus \((b, a) \in R\).

R is transitive because if \((a, b) \in R\) and \((b, c) \in R\), then \(a \equiv b \pmod{3}\) and \(b \equiv c \pmod{3}\), therefore \(a \equiv c \pmod{3}\), thus \((a, c) \in R\).

(e) \(R\) is not reflexive because e.g. \((1, 1) \notin R\) since \(1 \neq 1\).

R is not symmetric because e.g. \((2, 1) \in R\) but \((1, 2) \notin R\) since \(2 > 1\) but \(1 \neq 2\).

R is transitive because if \((a, b) \in R\) and \((b, c) \in R\), then \(a > b\) and \(b > c\), then \(a > c\), thus \((a, c) \in R\).