

MATH 114

Solution to problem 4.1 #18

How many positive integers less than 1000

Note: so we consider the integers 1, 2, ..., 999.

(a) are divisible by 7?

$$\lfloor \frac{999}{7} \rfloor = 142$$

(b) are divisible by 7 but not by 11?

As shown in (a), 142 integers are divisible by 7. Similarly, $\lfloor \frac{999}{77} \rfloor = 12$ integers are divisible by 77 (equivalently, are divisible by both 7 and 11). Therefore $142 - 12 = 130$ integers are divisible by 7 but not by 11.

(c) are divisible by bot 7 and 11?

12 - see part (b).

(d) are divisible by either 7 or 11?

Again, 142 integers are divisible by 7. Similarly, $\lfloor \frac{999}{11} \rfloor = 90$ integers are divisible by 11. 12 are divisible by both. Using the inclusion-exclusion formula on page 308, we see that $142 + 90 - 12 = 220$ integers are divisible by 7 or 11.

(e) are divisible by exactly one of 7 and 11?

$220 - 12 = 208$ (exclude the integers divisible by both 7 and 11 from the set of integers divisible by either 7 or 11).

(f) are divisible by neither 7 nor 11?

$999 - 220 = 779$ (the total number of integers minus the number of integers divisible by either 7 or 11).

(g) have distinct digits?

Count 1-digit, 2-digit, and 3-digit numbers separately.

1-digit integers: 1, ..., 9 - there are 9 integers.

2-digit integers: 10, ..., 99 - there are 90 integers, but in 9 of them (11, ..., 99) the two digits are the same. So there are $90 - 9 = 81$ 2-digit integers with distinct digits. Another way to get this answer is to consider the number of possibilities for each digit: the first digit can be any non-zero digit, so it has 9 choices. The second digit can be any digit except equal to the first one, so it has 9 choices too. There are $9 \cdot 9 = 81$ choices total.

3-digit integers: 100, ..., 999 - there are 900 integers total, but some of them have a repeating digit... The number of 3-digit integers with distinct digits can be counted as follows: the first digit can be any non-zero digit, so it has 9 choices. The second digit can be any digit except equal to the first one, so it has 9 choices too. Finally, the third digit can be any digit except equal to the first digit or the second digit, so it has 8 choices. There are $9 \cdot 9 \cdot 8 = 648$ choices total.

So there are $9 + 81 + 648 = 738$ positive integers less than 1000 with distinct digits.

(h) have distinct digits and are even?

Again, count 1-digit, 2-digit, and 3-digit numbers separately.

1-digit integers: the even ones are 2, 4, 6, 8. There are 4 of them.

2-digit integers: 10, 12, ..., 98 (45 of these), but in 4 of them (22, 44, 66, 88) the two digits are the same. So there are $45 - 4 = 41$ even 2-digit integers with distinct digits. Here is another way to think about this part: the first digit can be odd or even. If it is odd, then it has 5 choices (1, 3, 5, 7, 9) and the second digit can be any even digit, so it has 5 choices too. So $5 \cdot 5 = 25$ choices total so far. If the first digit is even, then it has 4 choices (2, 4, 6, 8), and the second digit can be any even digit except the same as the first one, so it has 4 choices again. We have $4 \cdot 4 = 16$ choices total in this case, so $25 + 16 = 41$ even 2-digit integers with distinct digits.

3-digit integers: The first digit can be odd or even, and the second digit can be odd or even. The third digit must be even. We have 4 cases, and will consider each case. Also, the first digit can be any non-zero digit, the second digit can be any digit except equal to the first digit, and the third digit can be any digit except equal to the first digit or the second digit. So the number of integers satisfying these conditions in each case is as follows:

Case 1. odd, odd, even: $5 \cdot 4 \cdot 5 = 100$

Case 2. odd, even, even: $5 \cdot 5 \cdot 4 = 100$

Case 3. even, odd, even: $4 \cdot 5 \cdot 4 = 80$

Case 4. even, even, even: $4 \cdot 4 \cdot 3 = 48$

The total number of even 3-digit integers with distinct digits is $100 + 100 + 80 + 48 = 328$.

So the total number of even integers less than 1000 with distinct digits is $4 + 41 + 328 = 373$.