MATH 114 Final Exam - Solutions

1. Is the function $f(x) = x^2$ from $\mathbb{N}$ to $\mathbb{N}$

   (a) one-to-one?
   Yes. Suppose $f(x_1) = f(x_2)$. For real numbers we have $x_1^2 = x_2^2 \rightarrow x_1 = \pm x_2$. Since x_1 and x_2 are natural numbers, $x_1 = x_2$.

   (b) onto?
   No. For example, 2 is not in the image because there is no $x \in \mathbb{N}$ such that $x^2 = 2$.

2. Use Mathematical Induction to prove that $2^n < n!$ for every positive integer n with $n \geq 4$.

   Basis step. If $n = 4$, $2^4 = 16 < 24 = 4!$.

   Inductive step. Assume $2^k < k!$. Then $2^{k+1} = 2^k \cdot 2 < k! \cdot 2 < k!(k+1) = (k+1)!$.

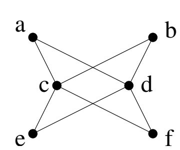
3. Let $P(x, y)$ denote the proposition $y = x + 5$ where x and y are positive integers. Determine the truth value of the following propositions.

   (a) $\forall x \exists y P(x, y)$
   True. For any positive integer x, $x + 5$ is a positive integer, so we can choose $y = x + 5$.

   (b) $\forall y \exists x P(x, y)$
   False. Counterexample: if $y = 1$, there is no positive integer x that satisfies $1 = x + 5$.

   (c) $\exists y \forall x P(x, y)$
   False. We will show that $\neg \exists y \forall x P(x, y)$ is true.
   $\neg \exists y \forall x P(x, y) \equiv \forall y \neg \forall x P(x, y) \equiv \forall y \exists x \neg P(x, y) \equiv \forall y \exists x \; y \neq x + 5$.
   For any positive integer y we can choose $x = y + 4$. Then $y = x - 4$, so $y \neq x + 5$.

4. Consider the following graph.



   (a) How many vertices does this graph have?
   6

   (b) How many edges does this graph have?
   8

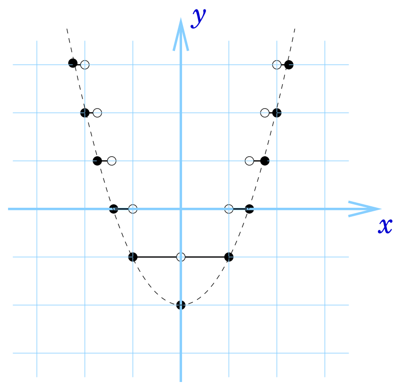
   (c) Is this graph bipartite?
   Yes. Label the vertices as shown above. The set of vertices can be partitioned into the following 2 sets: $V_1 = \{a, b, e, f\}$ and $V_2 = \{c, d\}$. Then every edge connects a vertex in V_1 and a vertex in V_2.

5. How many different strings can be made by reordering the letters of the word *SUCCESS*?

   $\frac{7!}{3!2!1!1!} = \frac{5040}{12} = 420$ *(by a theorem in the book) because there are three Ss, two Cs, one U, and one E.*

   Another way: there are $\binom{7}{3}$ ways to choose 3 positions for Ss, then $\binom{4}{2}$ ways to choose 2 positions for Cs, then 2 ways to choose the position for U, and the remaining letter must be E. There are $\binom{7}{3}\binom{4}{2}2 = \frac{7!}{3!4!}\frac{4!}{2!2!}2 = 35 \cdot 6 \cdot 2 = 420$.

6. Draw the graph of $f(x) = \lceil x^2 - 2 \rceil$.



7. (a) If $a|c$ and $b|c$, does a necessarily divide b?

   No. Counterexample: $a = 2$, $b = 3$, $c = 6$. *Then* $2|6$, $3|6$, *but* $2 \nmid 3$.

   (b) If $a|b$ and $b|c$, does a necessarily divide c?

   Yes. If $a|b$ *and* $b|c$ *then* $b = an$ *and* $c = bm$ *for some* $n, m \in \mathbb{Z}$. *Then* $c = bm = anm$ *and* $nm \in \mathbb{Z}$. *Therefore* $a|c$.

8. (a) Show that the relation $R = \{(a, b) \mid \lfloor a \rfloor = \lfloor b \rfloor\}$ on the set of real numbers is an equivalence relation.

   R is reflexive because for any $a \in \mathbb{R}$, $\lfloor a \rfloor = \lfloor a \rfloor$, *so* $(a, a) \in R$.

   R is symmetric because if $\lfloor a \rfloor = \lfloor b \rfloor$ *then* $\lfloor b \rfloor = \lfloor a \rfloor$, *so* $(a, b) \in R \rightarrow (b, a) \in R$.

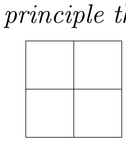
   R is transitive because if $\lfloor a \rfloor = \lfloor b \rfloor$ *and* $\lfloor b \rfloor = \lfloor c \rfloor$ *then* $\lfloor a \rfloor = \lfloor c \rfloor$, *so* $(a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$.

   (b) How many equivalence classes are there for this equivalence relation? Describe them.

   Each equivalence is an interval of the form $[n, n+1)$ *for some integer* n. *There are infinitely many equivalence classes:* $\ldots$, $[-1, 0)$, $[0, 1)$, $[1, 2)$, $[2, 3)$, $\ldots$.

9. Thirteen small insects are placed inside a 1×1 square. Show that at any moment there are at least four insects which can be covered by a single disk of radius $\dfrac{2}{5}$.

Divide the square into 4 smaller ($\frac{1}{2} \times \frac{1}{2}$) *squares as shown below. Think of insects as "objects" and small squares as "boxes" (note: assign each boundary point to one of the squares whose boundary it belongs to; it doesn't matter which one; for example, we could number the small squares and assign each boundary point to the square with the smallest number). Since there are 13 insects and 4 small squares, by the generalized Dirichlet's principle there is at least one square that contains at least* $\left\lceil \dfrac{13}{4} \right\rceil = 4$ *insects.*



Next we will show that a square $\frac{1}{2} \times \frac{1}{2}$ *can be covered by a disk of radius* $\dfrac{2}{5}$. *The diagonal of the square is* $\sqrt{\dfrac{1}{4} + \dfrac{1}{4}} = \sqrt{\dfrac{2}{4}} = \dfrac{\sqrt{2}}{2}$ *and the diameter of the disk is* $\dfrac{4}{5}$.

Since $50 < 64$, *we have* $\sqrt{50} < 8$, *or* $5\sqrt{2} < 8$. *Dividing both sides by 10 gives* $\dfrac{\sqrt{2}}{2} < \dfrac{4}{5}$.

We see that the diagonal of the square is less than the diameter of the disk, therefore the square can by covered by the disk.

Now, we showed that at least 4 insects are contained in one square, and the square can be covered by a disk, therefore at least 4 insects can be covered by a disk.

10. Prove that infinitely many Fibonacci numbers are divisible by 10.

Recall that Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ *for* $n \geq 2$. *Since* $F_n \equiv F_{n-1} + F_{n-2} (mod\ 10)$, *the last digit of each Fibonacci number* F_n *depends only on the last digits of two preceeding Fibonacci numbers. Let* D_n *be the last digit of* F_n *for each* $n \geq 0$. *Consider pais of consecutive last digits:* (D_0, D_1), (D_1, D_2), (D_2, D_3), *and so on. Since there are only 100 possible pairs of digits (there are 10 possibilities for each of the two digits), there must be a repetition among pairs* (D_k, D_{k+1}). *So for some nonnegative integers* l *and* m, $D_l = D_m$ *and* $D_{l+1} = D_{m+1}$. *In other words, there are two pairs of consecutive Fibonacci numbers with the same last digits. It follows that the last digits of the numbers immediately following these pairs are also the same, and the last digits of next numbers are the same, and so on. So the sequence of the last digits is periodic. Now calculate the last digits until we have a repetition described above:*

0, 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0, 9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1, **0, 1,** *1, 2, 3, 5,*

Since 0, 1 repeat, the sequence will repeat afterward. There are 0s in the cycle, so infinitely many Fibonacci numbers end with 0.

Note: actually, notice that the repetition must start with 0 and 1. Because if it doesn't then consider the first time that two consecutive last digits repeat. Since $F_{n-2} = F_n - F_{n-1}$, *the last digit of a Fibonacci number is determined by the last digits of the two numbers following it. Therefore the last digits of the numbers immediately preceeding the repeated pais are also the same. Contradiction. Hence 0, the last digit of* F_0, *is a part of the cycle. So we didn't even have to compute the above digits.*