

MATH 114 - Dirichlet's principle - Solutions to hw problems

4.2 #2 Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

Since there are 26 letters (assuming we use the Latin alphabet), there are more students (or last names) than letters, therefore by Dirichlet's principle (think of students as objects and letters as boxes) there is at least one letter such that at least two last names start with that letter.

4.2 #8 Show that if f is a function from S to T where S and T are finite sets with $|S| > |T|$, then there are elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$, or in other words, f is not one-to-one.

We will use Dirichlet's principle. Think of elements of S as objects and elements of T as boxes. A function sends each element of S to an element of T (i.e. "puts an object into a box"). Since there are more elements of S (objects) than elements of T (boxes), there is at least one element of T that is the image of at least two elements of S .

4.2 #14 (a) Show that if seven integers are selected from the first 10 positive integers, there are must be at least two pairs of these integers with the sum 11.

Divide the first ten positive integers into the following 5 groups: $\{1, 10\}$, $\{2, 9\}$, $\{3, 8\}$, $\{4, 7\}$, $\{5, 6\}$. Notice that the sum of the two numbers in any group is equal to 11.

Proof 1. Since we choose 7 numbers and there are 5 (which is less than 7) groups, by Dirichlet's principle at least two numbers are in the same group. Their sum is 11. Now, besides these 2 numbers in one group, we chose 5 other numbers from 4 other groups. Again, there are more numbers than groups, therefore at least two numbers are in one group. These two numbers form another pair with sum equal to 11.

Proof 2. Since we choose 7 numbers out of 10, only 3 numbers remain unchosen. They can be in a maximum of 3 groups. Therefore at least 2 groups are chosen completely, so we have at least 2 pairs with sum equal to 11.

(b) Is the conclusion in part (a) true if six integers are selected rather than seven?

No. For example, if we choose 1, 2, 3, 4, 5, and 6, then there is only one pair with sum equal to 11: 5 and 6.

4.2 #16 How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

The answer is 5. Divide the above numbers into the following 4 groups: $\{1, 15\}$, $\{3, 13\}$, $\{5, 11\}$, $\{7, 9\}$. If we choose 5 numbers out of 4 groups, then by Dirichlet's principle we'll have at least 2 numbers in the same group, and their sum will be equal to 16. It is not sufficient to choose 4 numbers. For example, we could choose 1, 3, 5, 7, and the sum of no 2 of these is equal to 16.

4.2 #32 A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

Each computer can be connected to 1, 2, 3, 4, or 5 other computers. Since there are 6 computers, there are more computers than possible numbers of connections, by Dirichlet's principle at least two computers that have the same number of connections.

4.2 #40 There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.

Divide the addresses from 1000 to 1099 (inclusive) into the following 50 groups: $\{1000, 1001\}$, $\{1002, 1003\}$, ... , $\{1098, 1099\}$. Since there are more houses (or addresses) than groups, by Dirichlet's principle at least two addresses are in the same group. They are consecutive.