MATH 114 Homework 13 & 14 - Selected solutions

Nobody has more than 300,000 hairs on his head. A city has 300,001 inhabitants. Can you assert with certainty that there are two persons in this city with the same number of hairs on their heads?

No, because one person can be bald, another person can have 1 hair, third person can have 2 hairs, ..., 300,001st person can have 300,000 hairs. So all inhabitants may have different number of hairs.

Note: Dirichlet's principle doesn't apply here because the number of "objects" (people) is equal to the number of "boxes" (possibilities for the number of hairs: 0 through 300,000).

Show that from 52 positive integers, we can select two such that their sum or difference is divisible by 100. Is the assertion also valid for 51 positive integers?

Let's consider the remainder of each of the 52 integers. We have 52 remainders. Each remainder is a nonnegative number less than 100.

Divide the possible remainders into the following 51 sets: $\{0\}, \{1, 99\}, \{2, 98\}, \ldots, \{49, 51\}$, $\{50\}$ (there are 49 pairs and 2 sets containing just one element).

Think of the 52 remainders of the given numbers as "objects" and the 51 sets of possible remainders as "boxes". Since we have more remainders than sets, by Dirichlet's principle at least 2 remainders are in the same set. These 2 remainders can be either equal or different.

If two remainders are equual then the difference of the corresponding numbers is divisible by 100 (because we know that $a - b$ is divisible by an integer n if and only if a and b have the same remainder when divided by n).

If two remainders are in the same set and are different, then they add up to 100 (since $1+99 = 2+98 = \ldots = 49+51 = 100$), then the sum of the corresponding numbers is divisible by 100: If $a = 100q + r$, $b = 100s + t$ and $r + t = 100$, then $a + b = 100q + r + 100s + t = 100q + 100t + 100 = 100(q + t + 1)$.

Therefore either the difference or the sum of the two numbers whose remainders are in the same set is divisible by 100.

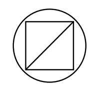
The statement is false for 51 positive integers. Counterexample: 1, 2, 3, 4, ..., 49, 50, 100. The largest possible difference is 99 which is less than 100, so no difference is divisible by 100. The sum of 2 numbers not exceeding 50 is at most 99 and hence is not divisible by 100. Finally, the sum of 100 and any other number from this set is not divisible by 100.

Note: the proof above does not work for 51 numbers because then the number of "objects" is equal to the number of "boxes".

Suppose that fifty-one small insects are placed inside a square of side 1. Show that at any moment there are at least three insects which can be covered by a single disk of radius 1/7.

Let's divide the square into 25 small squares, each with side 1/5, as shown on the picture below. Think of insects as "objects" and small squares as "boxes" (note: assign each boundary point to one of the squares whose boundary it belongs to; it doesn't matter which one; for example, we could number the small squares and assign each boundary point to the square with the smallest number). Since there are 51 insects and 25 small squares, by the generalized Dirichlet's principle there is at least one square that contains at least $\left\lceil \frac{51}{25} \right\rceil = 3$ insects.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25



Next we will show that a square with side 1/5 can be covered by a disk of radius 1/7. The diagonal of the square is $\sqrt{\frac{1}{25} + \frac{1}{25}} = \sqrt{\frac{2}{25}} = \frac{\sqrt{2}}{5}$ and the diameter of the disk is $\frac{2}{7}$.

Since $98 < 100$, we have $\sqrt{98} < 10$, or $7\sqrt{2} < 10$. Dividing both sides by 35 gives $\frac{\sqrt{2}}{5} < \frac{2}{7}$.

We see that the diagonal of the square is less than the diameter of the disk, therefore the square can by covered by the disk.

Now, we showed that at least 3 insects are contained in one square, and the square can be covered by a disk, therefore at least 3 insects can be covered by a disk.

7.5 #10. Let R be the relation on the set of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

For any a and b, $ab = ba$, so $((a,b),(a,b)) \in R$. Hence R is reflexive.

If $((a,b),(c,d)) \in R$ then $ad = bc$, or $cb = da$, so $((c,d),(a,b)) \in R$. Hence R is symmetric.

If $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$ then $ad = bc$ and $cf = de$. Solving the first equation for a and the second equation for f gives $a = \frac{bc}{d}$ and $f = \frac{de}{c}$. Then $af = \frac{bcde}{dc} = be$, so $((a,b),(e,f)) \in R$. Hence R is transitive.

Therefore R is an equivalence relation.

7.5 #20(c). What are the equivalence classes of the equivalence relation $\{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3)\}$ on the set $\{0,1,2,3\}$?

Since 0 is only equivalent to itself, 1 and 2 are equivalent to themselves and each other, and 3 is only equivalent to itself, we have $[0] = \{0\}$, $[1] = \{1, 2\}$, $[2] = \{1, 2\}$, and $[3] = \{3\}$.

So there are 3 equivalence classes: $\{0\}$, $\{1, 2\}$, and $\{3\}$.

Let $|A| = n$. How many
(a) symmetric
(b) both symmetric and antisymmetric
(c) reflexive, symmetric, and antisymmetric
relations are there on A?

First of all, recall that a relation on A is a subset of $A \times A = \{(a, b)|a, b \in A\}$. Thus the number of certain relations is the number of subsets of $A \times A$ that satisfy the corresponding properties.

(a) A relation R is called symmetric if for any $(a, b) \in R$, we also have $(b, a) \in R$. Notice that this condition gives no restrictions on pairs of the form (a, a).

There are n pairs of the form (a, a) where $a \in A$, and each such pair may or may not be in R. Hence we have 2^n choices for such pairs.

The remaining $n^2 - n$ pairs can be divided into groups of two of the form $\{(a, b),(b, a)\}$ where $a \neq b$. For each such group, either both pairs (a, b) and (b, a) are in R or neither pair is in R. Since there are $\dfrac{n^2 - n}{2}$ such groups, we have $2^{\frac{n^2-n}{2}}$ choices for pairs of these form.

There are $2^n 2^{\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}}$ choices total, so $2^{\frac{n^2+n}{2}}$ symmetric relations.

(b) A relation R is called antisymmetric if whenever $(a, b) \in R$ and $(b, a) \in R$, $a = b$. Again, this condition gives no restrictions on pairs of the form (a, a), so we have 2^n choices for such pairs.

No pair of the form (a, b) where $a \neq b$ can be in R because if it were then since R is symmetric, (b, a) would be in R. Since R is also antisymmetric, $a = b$. This contradicts to the condition $a \neq b$. So there is only one possibility: none of the pairs (a, b) with $a \neq b$ is in R.

Thus we have 2^n relations.

(c) R is called reflexive if for every $a \in A$, $(a, a) \in R$. So all pairs of the form (a, a) must be in R. In part (b) we showed that no pair of the form (a, b), $a \neq b$ can be in R, so there is only one relation that satisfies all these conditions, namely, $\{(a, a)|a \in A\}$.