8.2 # 18(f) (optional)

Let the cube be the set of points \((x_1, x_2, x_3, x_4)\) in \(\mathbb{R}\) such that \(0 \leq x_i \leq 1\) for each \(i\). Then the set of vertices is the set of ordered 4-tuples of zeros and ones. There are 16 vertices. Two vertices are connected if and only if their coordinates differ in exactly one position. The set of vertices with \(x_4 = 0\) and edges connecting these vertices form \(Q_3\) shown in black. The vertices with \(x_4 = 1\) and edges connecting these form another copy of \(Q_3\), shown in blue. Finally, red edges connect vertices in the black \(Q_3\) and the blue \(Q_3\) whose coordinates \(x_1, x_2, x_3\) are the same (but \(x_4\) are different).

8.2 #24 For which values of \(n\) are these graphs bipartite?

(a) \(K_n\)

\(K_1\) is bipartite if we allow one of the sets \((V_1\) or \(V_2\) using the notation in definition 5 on page 550) to be empty (the book does).

\(K_2\) is bipartite because we can let one vertex be in \(V_1\) and the other vertex to be in \(V_2\).

\(K_n\) for \(n \geq 3\) is not bipartite: choose any 3 vertices. They all are pairwise connected, therefore there is no way to partition them into two disjoint sets \(V_1\) or \(V_2\) such that there are no edges within \(V_1\) and no edges within \(V_2\).

(b) \(C_n\)

\(C_n\) is bipartite if and only if \(n\) is even. Label the vertices by 1, 2, ..., consecutively along the cycle. If vertex 1 is in \(V_1\) then vertex 2 must be in \(V_2\), vertex 3 must be in \(V_1\), vertex 4 must be in \(V_2\), and so on. All vertices with odd number are in \(V_1\) and all vertices with even number are in \(V_2\). The last vertex is in \(V_1\) if \(n\) is odd and it is in \(V_2\) if \(n\) is even. But it is connected to vertex 1. We see that if \(n\) is odd, the graph is not bipartite, and if \(n\) is even, the graph is bipartite.

(d) \(Q_n\) (optional)

\(Q_n\) is bipartite for any \(n\). Let \(V_1\) consist of all vertices whose sum of coordinates is odd and let \(V_2\) consist of all vertices whose sum of coordinates is even. Two vertices in \(Q_n\) are connected if and only if their coordinates differ in only one position, therefore the sums of their coordinates have different parity, so they are in different sets.