MATH 114 Fall 2004

Solutions to practice problems for Final Exam

Reminder: the **final exam** is on Monday, December 13 from 11am - 1am.

Office hours: Thursday, December 9 from 1 - 5pm; Friday, December 10 from 11am - 3pm.

1. (1.1, 1.2) Each inhabitant of a remote village always tells the truth or always lies. A villager will only give a "yes" or "no" response to a question a tourist asks. Suppose you are a tourist visiting this area and come to a fork in the road. One branch leads to the ruins you want to visit; the other branch leads deep into the jungle. A villager is standing at the fork in the road. Explain how the villager's answer to your question "if I were to ask you whether the right branch leads to the ruins, would you answer yes?" will tell you which road to take. Which logic law are you using?

   Let question 1 be "Does the right branch lead to the ruins?" and let question 2 be "if I were to ask you whether the right branch leads to the ruins, would you answer yes?".
   Let's consider each case:

	the right road leads to the ruins	*what the villager would have answered to question 1*	*the villager's answer to question 2*
This villager always tells the truth	*yes* *no*	*yes* *no*	*yes* *no*
This villager always lies	*yes* *no*	*no* *yes*	*yes* *no*

   We see that if the villager always tells the truth then he always tells you correctly whether or not you should take the right branch.
   If the villager always lies then when he hears your question 2, he knows he would have lied to question 1, but he lies again, so in the end he again tells you correctly whether or not you should take the right branch. We are using the double negation law.

2. (1.2) Use a truth table to show that propositions $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	**T**	T	T	**T**
T	T	F	F	**T**	T	T	**T**
T	F	T	F	**T**	T	T	**T**
T	F	F	F	**T**	T	T	**T**
F	T	T	T	**T**	T	T	**T**
F	T	F	F	**F**	T	F	**F**
F	F	T	F	**F**	F	T	**F**
F	F	F	F	**F**	F	F	**F**

   Since the 5th and the 8th columns (in bold) are the same, the propositions are logically equivalent.

3. (1.3) Translate the statement

$$\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$$

   into English, where $C(x)$ is "x has a computer", $F(x, y)$ is "x and y are friends", and the domain for both x and y is the set of all students in Fresno.

   Every student in Fresno either has a computer or has a friend who has a computer.

4. (1.4) Let $P(x, y)$ denote the proposition "$x < y$" where x and y are real numbers. Determine the truth values of

   (a) $\exists x \exists y P(x, y)$,

   True. Example: $x = 1$, $y = 2$.

   (b) $\forall x \exists y P(x, y)$,

   True. For any x *we can choose* $y = x + 1$. *Then* $x < y$.

   (c) $\exists x \forall y P(x, y)$,

   False. No matter what x *is, the proposition* $x < y$ *is not true for all* y. *For example, it is false for* $y = x - 1$.
   Another way to show that $\exists x \forall y P(x, y)$ *is false is to show that* $\neg \exists x \forall y P(x, y)$ *is true.*
   $\neg \exists x \forall y P(x, y) = \forall x \neg \forall y P(x, y) = \forall x \exists y \neg P(x, y) = \forall x \exists y \; x \geq y$.
   This is true because for any x *we can choose* $y = x - 1$ *and then* $x \geq y$.

   (d) $\forall x \forall y P(x, y)$.

   False. Counterexample: $x = 1$, $y = 0$.
   Another way to show that $\forall x \forall y P(x, y)$ *is false is to show that* $\neg \forall x \forall y P(x, y)$ *is true.*
   $\neg \forall x \forall y P(x, y) = \exists x \neg \forall y P(x, y) = \exists x \exists y \neg P(x, y) = \exists x \exists y \; x \geq y$.
   Example: $x = 2$, $y = 1$.

5. (1.5) Show that $\sqrt[3]{25}$ is irrational.

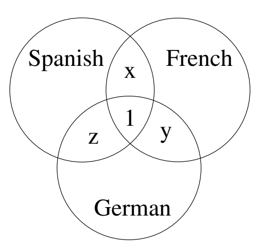
   Suppose $\sqrt[3]{25}$ *is rational. Then* $\sqrt[3]{25} = \dfrac{p}{q}$ *for some integers* p *and* q *with* $(p, q) = 1$. *Then* $25 = \dfrac{p^3}{q^3}$, *or* $25q^3 = p^3$. *Since the left hand side is divisible by 5,* p^3 *is divisible by 5. Since 5 is prime,* p *is divisible by 5. Therefore* $p = 5n$ *for some integer* n, *and* $p^3 = 125n^3$. *Then* $25q^3 = 125n^3$, *or* $q^3 = 5n^3$. *Now* q^3 *is divisible by 5, therefore* q *is divisible by 5, which contradicts to the condition* $(p, q) = 1$. *So our assumption that* $\sqrt[3]{25}$ *is rational is false.*

6. (1.6) If $A = \{1, 2, 3\}$, $B = \{a, b\}$, and $C = \emptyset$, list all elements of $A \times B$ and $A \times C$.

   $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$ *and* $A \times C$ *is empty because* C *is empty.*

7. (1.7) In a group of 20 people 14 speak Spanish, 9 speak French, and 4 speak German. Every person speaks at least one of these languages but only one person speaks all three. How many people speak exactly 2 of these languages?

   Draw a Venn diagram:

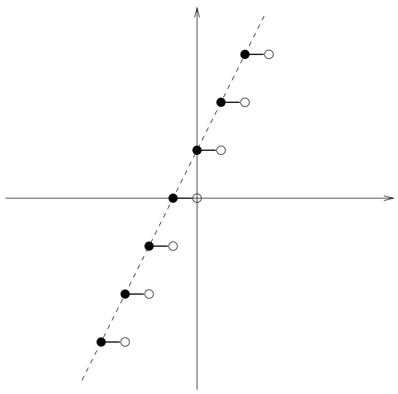


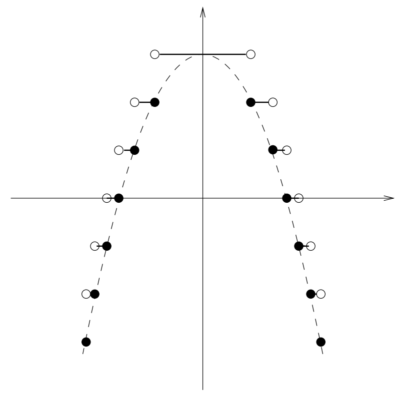
Let x be the number of people who speak Spanish and French but not German, let y be the number of people who speak French and German but not Spanish, and let z be the number of people who speak Spanish and German but not French. Then the total number of people is

$$20 = 14 + 9 + 4 - x - y - z - 2$$

(we have to subtract x, y, and z because we counted these people twice when adding $14+9+4$, and we have to subtract 2 because we counted the person who speaks all three languages 3 times.) We have $20 = 25 - x - y - z$, or $x + y + z = 5$. So 5 people speak exactly 2 of these languages.

8. (1.8) Draw the graphs of $f(x) = \lfloor 2x + 1 \rfloor$ and $g(x) = \lceil 3 - x^2 \rceil$. If we consider $f(x)$ and $g(x)$ as functions from $\mathbb{R}$ to $\mathbb{Z}$, are they one-to-one? Are they onto?





Neither function is one-to-one because for example $f(0) = f(0.1)$ and $g(0) = f(0.1)$.
f is onto: for any $y \in \mathbb{Z}$, there exists $x \in \mathbb{R}$ such that $\lfloor 2x + 1 \rfloor = y$. For example, we can choose $x = \dfrac{y-1}{2}$. Then $\lfloor 2x + 1 \rfloor = \lfloor 2\frac{y-1}{2} + 1 \rfloor = \lfloor y \rfloor$.
g is not onto: for example, there is no x such that $g(x) = 4$ because $x^2 \geq 0$, so $3 - x^2 \leq 3$, so $\lceil 3 - x^2 \rceil \leq 3$.

9. (2.4)

   (a) If $a|b$ and $a|c$, does a necessarily divide $b + c$?

   Yes. If $a|b$ and $a|c$ then $b = an$ and $c = am$ for some integer n and m. Then $b + c = an + am = a(n + m)$. Since $n + m \in \mathbb{Z}$, $a|(b + c)$.

   (b) If $a|c$ and $b|c$, does $a + b$ necessarily divide c?

   No. Counterexample: if $a = 2$, $b = 3$, and $c = 6$, then $2|6$, $3|6$, but $2 + 3 = 5 \not| 6$.

10. (2.5)

    (a) Convert 12 from decimal notation to binary notation.

    Since $12 = 8 + 4 = 2^3 + 2^2 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$, $12 = (1100)_2$.

    (b) Convert 68DAC0 from hexadecimal (base 16) notation to decimal notation.

    $(68DAC0)_{16} = 6 \cdot 16^5 + 8 \cdot 16^4 + 13 \cdot 16^3 + 10 \cdot 16^2 + 12 \cdot 16^1 + 0 \cdot 16^0 = 6871744$.

11. (2.6) Find the greatest common divisor d of $a = 46$ and $b = 32$, and integer numbers x and y such that $xa + yb = d$.

    $46 = 1 \cdot 32 + 14$
    $32 = 2 \cdot 14 + 4$
    $14 = 3 \cdot 4 + 2$
    $4 = 2 \cdot 2 + 0$
    So $d = 2$. *Solving for* d *from the third equation and working upward, we have* $2 = 14 - 3 \cdot 4 = 14 - 3(32 - 2 \cdot 14) = 14 - 3 \cdot 32 + 6 \cdot 14 = 7 \cdot 14 - 3 \cdot 32 = 7(46 - 1 \cdot 32) - 3 \cdot 32 = 7 \cdot 46 - 7 \cdot 32 - 3 \cdot 32 = 7 \cdot 46 - 10 \cdot 32$. *So* $x = 7$ *and* $y = -10$.

12. (3.1) Show that 3 divides $a^2 + b^2$ iff 3 divides both a and b (where a and b are integers).

    Consider all possible remainders of a *and* b *when divided by 3:*

	$a \equiv 0(mod\ 3)$ $a^2 \equiv 0(mod\ 3)$	$a \equiv 1(mod\ 3)$ $a^2 \equiv 1(mod\ 3)$	$a \equiv 2(mod\ 3)$ $a^2 \equiv 1(mod\ 3)$
$b \equiv 0(mod\ 3)$ $b^2 \equiv 0(mod\ 3)$	$a^2 + b^2 \equiv 0(mod\ 3)$	$a^2 + b^2 \equiv 1(mod\ 3)$	$a^2 + b^2 \equiv 1(mod\ 3)$
$b \equiv 1(mod\ 3)$ $b^2 \equiv 1(mod\ 3)$	$a^2 + b^2 \equiv 1(mod\ 3)$	$a^2 + b^2 \equiv 2(mod\ 3)$	$a^2 + b^2 \equiv 2(mod\ 3)$
$b \equiv 2(mod\ 3)$ $b^2 \equiv 1(mod\ 3)$	$a^2 + b^2 \equiv 1(mod\ 3)$	$a^2 + b^2 \equiv 2(mod\ 3)$	$a^2 + b^2 \equiv 2(mod\ 3)$

    We see that $a^2 + b^2 \equiv 0(mod\ 3)$ *only in one case:* $a \equiv 0(mod\ 3)$ *and* $b \equiv 0(mod\ 3)$.

13. (3.2) Find the value of the sum $\displaystyle\sum_{i=1}^{100}(2i^2 + 2^i)$.

$$\sum_{i=1}^{100}(2i^2 + 2^i) = \sum_{i=1}^{100} 2i^2 + \sum_{i=1}^{100} 2^i = 2\sum_{i=1}^{100} i^2 + (2 + 2^2 + 2^3 + 2^4 + \ldots + 2^{100})$$
$$= 2\frac{100 \cdot 101 \cdot 201}{6} + 2(1 + 2 + 2^2 + \ldots + 2^{99}) = \frac{100 \cdot 101 \cdot 201}{3} + \frac{2(1 - 2^{100})}{1 - 2}$$
$$= 10100 \cdot 67 - 2 + 2^{101} = 676700 - 2 + 2^{101} = 676698 + 2^{101}.$$

14. (3.3) Prove that $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$ whenever n is a positive integer greater than 1.

    We will prove this inequality by Mathematical Induction.
    Basis step: if $n = 2$ *then* $1 + \frac{1}{4} < 2 - \frac{1}{2}$ *is true since* $\frac{5}{4} < \frac{3}{2}$.
    Inductive step: assume $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} < 2 - \frac{1}{k}$ *is true. We want to prove that* $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$. *Using the inductive hypothesis we have*

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{(k+1)^2} = \left(1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2}\right) + \frac{1}{(k+1)^2}$$
$$< 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \frac{(k+1)^2 - k}{k(k+1)^2} = 2 - \frac{k^2 + k + 1}{k(k+1)^2}$$
$$< 2 - \frac{k^2 + k}{k(k+1)^2} = 2 - \frac{k(k+1)}{k(k+1)^2} = 2 - \frac{1}{k+1}.$$

    (Note: to find this chain of equalities and inequalities, work form left to right until you bring 2 fractions to a common denominator and simplify the numerator, and work from right (what you want to get) to left making the denominator of the fraction the same as the denominator on the left. Then notice that the inequality holds. Now you have a complete proof.)

15. (3.4) Let $f_1(x) = 2x + 1$ and $f_n = f_1 \circ f_{n-1}$. Compute f_n for some small values of n. Notice the pattern. Write a formula for f_n and prove it using Mathematical Induction.

    $f_1(x) = 2x + 1$
    $f_2(x) = 2(2x + 1) + 1 = 4x + 3$
    $f_3(x) = 2(4x + 3) + 1 = 8x + 7$
    $f_4(x) = 2(8x + 7) + 1 = 16x + 15$
    We notice that the coefficient of x is a power of 2 and the free term is a power of 2 minus 1.
    Guess: $f_n(x) = 2^n + (2^n - 1)$.
    Proof: for $n = 1$ the formula holds.
    Inductive step: assume $f_k(x) = 2^k + (2^k - 1)$. *Then* $f_{k+1}(x) = 2f_k(x) + 1 = 2(2^k x + 2^k - 1) + 1 = 2^{k+1}x + 2^{k+1} - 2 + 1 = 2^{k+1}x + (2^{k+1} - 1)$.

16. (4.1) A witness to a hit-and-run accident tells the police that the license plate of the car in the accident, which contains three letters followed by three digits, starts with the letters AS and contains both the digits 1 and 2. How many different license plates can fit this description?

    We have 26 choices for the third letter.
    If the third digit is not 1 or 2 then there are 8 choices for it, and the number of permutations of 1, 2, and another digit is $3! = 6$. *So we have 48 possibilities here. If the third digit is 1, there are 3 possibilities: 112, 121, and 211. Similarly, there are 3 possibilities if the third digit is 2. Thus the total number of possibilities for the digits is* $48 + 3 + 3 = 54$. *The total number of license plates is then* $26 \cdot 54 = 1404$.

17. (4.2) How many people are needed to guarantee that at least six of them have the same sign of the zodiac?

    If there are N people then by the generalized Dirichlet's principle at least $\left\lceil \frac{N}{12} \right\rceil$ *of them have the same sign of the zodiac. So we need N to satisfy* $\left\lceil \frac{N}{12} \right\rceil \geq 6$, *or* $\frac{N}{12} > 5$, *or* $N > 60$. *Thus we need at least 61 people. If there are less than or equal to 60 people then it is possible that at most 5 have the same sign of the zodiac (why?).*

18. (4.3) There are 10 projects and 5 groups of people.

    (a) How many ways are there to choose 5 projects out of 10?

    $$C(10, 5) = \binom{10}{5} = \frac{10!}{5!5!}$$

    (b) How many ways are there to choose 5 projects out of 10 and assign them to the 5 groups so that each group is assigned one project?

    $$P(10, 5) = \frac{10!}{5!}$$

    (c) How many ways are there to assign all 10 projects to the 5 groups so that each group is assigned two projects?

    There are $\binom{10}{2}$ *ways to choose 2 projects for the first group,* $\binom{8}{2}$ *ways to choose 2 projects for the second group,* $\binom{6}{2}$ *ways to choose 2 projects for the third group,* $\binom{4}{2}$ *ways to choose 2 projects for the fourth group, and the remaining 2 projects must be assigned to the last group. Thus there are* $\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}$ *ways to assign the projects.*

19. (4.4) Show that if n is a positive integer then $\sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n$.

By the Binomial Theorem, $4^n = (1+3)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 3^k = \sum_{k=0}^{n} 3^k \binom{n}{k}$.

20. (4.5) How many ways are there to travel in xyz space from the origin $(0,0,0)$ to the point $(3,4,5)$ by taking steps one unit in the positive x direction, one unit in the positive y direction, or one unit in the positive z direction? (Moving in the negative x, y, or z direction is prohibited.)

We need to make a total of 3 steps in the positive x direction, a total of 4 steps in the positive y direction, a total of 5 steps in the positive z direction. So there are 12 steps total. Each path can be described by a string of length 12 that has 3 xs, 4 ys, and 5 zs. By theorem 3 on page 340, the number of arrangements of these letters is $\frac{12!}{3!4!5!}$.

21. (7.1) Consider the relation R on the set of natural numbers defined by $(a,b) \in R$ iff $\log_a b \in \mathbb{Z}$. Is R reflexive? Symmetric? Antisymmetric? Transitive?

R is not reflexive because $\log_0 0$ is undefined.
R is not symmetric because e.g. $\log_2 4 \in \mathbb{Z}$ but $\log_4 2 \notin \mathbb{Z}$.
R is antisymmetric because if $\log_a b = n \in \mathbb{Z}$ and $\log_b a = m \in \mathbb{Z}$ then $1 = \log_a a = \log_a b \log_b a = nm$, so n and m are either both 1 or both -1. If $n = m = 1$ then $a = b$; if $n = m = -1$ then $a = \frac{1}{b}$, and since a and b are natural numbers, $a = b = 1$ (which is actually impossible since $\log_a b$ is only defined for $a \neq 1$).
If $\log_a b \in \mathbb{Z}$ and $\log_b c \in \mathbb{Z}$ then $\log_a c = \log_a b \cdot \log_b c \in \mathbb{Z}$, so R is transitive.

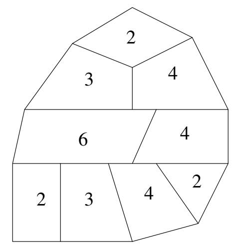
22. (7.5) Show that the relation R on the set of functions from $\mathbb{R}$ to $\mathbb{R}$ defined by $(f(x), g(x)) \in R$ iff $f(0) = g(0)$ is an equivalence relation. Describe its equivalence classes.

Since for any function $f(x)$, $f(0) = f(0)$, $(f,f) \in R$, so R is reflexive.
If $f(0) = g(0)$ then $g(0) = f(0)$, so $(f,g) \in R$ implies that $(g,f) \in R$, so R is symmetric.
If $f(0) = g(0)$ and $f(0) = h(0)$ then $f(0) = h(0)$, so $(f,g) \in R$ and $(g,h) \in R$ implies that $(f,h) \in R$, so R is transitive.
Therefore R is an equivalence relation. There are infinitely many equivalence classes, namely, for each $a \in R$, we have an equivalence class consisting of those functions f for which $f(0) = a$.

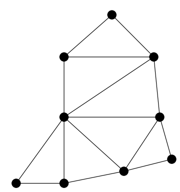
23. (8.1)
   (a) There are 9 counties in Sikinia. There are no "four corners" points (like Arizona, Colorado, New Mexico, and Utah). Each county counted the number of neighboring counties. The numbers are 5, 4, 4, 4, 3, 3, 2, 2, and 2. Prove that at least one county made a mistake.

   There exists a simple graph whose vertices represent counties and two vertices are connected if and only if the corresponding counties are neighbors. The degree of each vertex in this graph is the number of neighbors of the county this vertex represents. We know that in any graph the sum of the degrees of its vertices is even, but the sum of the given numbers is odd. Contradiction.

(b) If the numbers are actually 6, 4, 4, 4, 3, 3, 2, 2, and 2, draw a possible map.



(c) For the above map, draw a graph with vertices representing counties where two vertices connected if and only if the correponding counties are neighbors.



24. (8.2) There are 10 men and 10 women attending a dance. Each man knows exactly two women and each woman knows exactly two men. Show that after suitable pairing, each man can dance with a woman he knows.

There is a bipartite graph whose vertices represent these 20 people, and two vertices are connected if and only if the people they represent know each other. Choose any man. He knows exactly 2 women, consider any of them. She knows exactly 2 men, the one mentioned above and another one. He knows another woman, and so on. Sooner or later we will hit a woman who knows the first man, so we'll get a cycle like one shown below. Choosing consecutive men and women in this cycle we can pair them up so that in each pair the man and the woman know each other (pairs in our example are shown by bold edges). If there are any people left (not in this cycle), do this process again.