MATH 114
Test 1 - Solutions
October 1, 2004

1. Show that \((-q \land (p \rightarrow q)) \rightarrow \neg p\) is a tautology.

   There are many ways to prove this. One is using a truth table. Others use logical equivalences.
   One short proof I could find is as follows.
   \((-q \land (p \rightarrow q)) \rightarrow \neg p \equiv q \lor \neg p \equiv q \lor \neg p \equiv (p \lor q) \lor \neg(p \lor q) \equiv (p \lor q) \lor \neg(p \lor q) \equiv T

2. Let \(P(x, y)\) be the statement \(x^2 < y\) where both \(x\) and \(y\) are real numbers. Determine the
   truth values of the following propositions. Give examples and explanations to support your
   answers.

   (a) \(P(3, 4)\) is false because \(9 > 4\).
   (b) \(\forall x \forall y P(x, y)\) is false. Counterexample: \(x = 3, y = 4\), see (a).
   (c) \(\exists y \exists x P(x, y)\) is true. Example: \(x = 1, y = 2\).
   (d) \(\forall x \exists y P(x, y)\) is true. Given an \(x\), we can choose \(y = x^2 + 1\), and then \(x^2 < y\).
   (e) \(\exists x \forall y P(x, y)\) is false. No matter what \(x\) is, \(P(x, 0)\) is false.

3. Prove that the sum of two odd numbers is even.
   Let \(2n + 1\) and \(2m + 1\) be two odd numbers. Then their sum is \((2n + 1) + (2m + 1) =
   2n + 2m + 2 = 2(n + m + 1)\). It is divisible by 2, and therefore is even.

4. Let \(S = \{1, 2, 3, 4\}\) and \(T = \{2, 4, 5\}\). Find the following:

   (a) The cardinality of \(S\) is 4 (the number of elements in \(S\))
   (b) \(S \cup T = \{1, 2, 3, 4, 5\}\) (the union of \(S\) and \(T\))
   (c) \(S \cap T = \{2, 4\}\) (the intersection of \(S\) and \(T\))
   (d) \(S - T = \{1, 3\}\) (the set of elements of \(S\) which are not in \(T\))
   (e) How many elements are there in \(S \times T\)? 12 because \(S \times T\) consists of all pairs of the
      form (element of \(S\), element of \(T\)), and there are 4 elements in \(S\) and 3 elements in \(T\)

5. Which of the following functions \(\mathbb{R} \rightarrow \mathbb{R}\) are one-to-one? onto? Explain.

   (a) \(f(x) = -x + 2\) is both one-to-one and onto.
      One-to-one: \(f(x_1) = f(x_2)\) \(\Rightarrow -x_1 + 2 = -x_2 + 2\) \(\Rightarrow -x_1 = -x_2\) \(\Rightarrow x_1 = x_2\)
      Onto: Let \(y \in \mathbb{R}\), then \(f(-y + 2) = -(-y + 2) + 2 = y - 2 + 2 = y\)
   (b) \(f(x) = e^x\) is one-to-one but not onto.
      One-to-one: \(f(x_1) = f(x_2)\) \(\Rightarrow e^{x_1} = e^{x_2}\) \(\Rightarrow \ln e^{x_1} = \ln e^{x_2}\) \(\Rightarrow x_1 = x_2\)
      Not onto because there is no \(x\) such that \(e^x = 0\) (since \(e^x\) takes on only positive values)
   (c) \(f(x) = x^4\) is neither one-to-one nor onto.
      Not one-to-one: \(f(1) = f(-1)\) but \(1 \neq -1\)
      Not onto because there is no \(x\) such that \(x^4 = -1\)

6. Sketch the graphs of \(f(x) = |1 - x|\) and \(g(x) = |1 - x|\).