# MATH 114
# Test 1 - Solutions

October 1, 2004

1. Show that $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology.

   *There are many ways to prove this. One is using a truth table. Others use logical equivalences. One short proof I could find is as follows.*

   $(\neg q \wedge (p \to q)) \to \neg p \equiv \neg(\neg q \wedge (p \to q)) \vee \neg p \equiv q \vee \neg(p \to q) \vee \neg p \equiv (\neg p \vee q) \vee \neg(p \to q) \equiv (p \to q) \vee \neg(p \to q) \equiv T$

2. Let $P(x, y)$ be the statemnt $x^2 < y$ where both $x$ and $y$ are real numbers. Determine the truth values of the following propositions. Give examples and explanations to support your answers.

   (a) $P(3, 4)$ *is false because* $9 > 4$.
   (b) $\forall x \forall y P(x, y)$ *is false. Counterexample:* $x = 3$, $y = 4$, *see (a).*
   (c) $\exists y \exists x P(x, y)$ *is true. Example:* $x = 1$, $y = 2$.
   (d) $\forall x \exists y P(x, y)$ *is true. Given an* $x$, *we can choose* $y = x^2 + 1$, *and then* $x^2 < y$.
   (e) $\exists x \forall y P(x, y)$ *is false. No matter what* $x$ *is, P(x,0) is false.*

3. Prove that the sum of two odd numbers is even.

   *Let* $2n + 1$ *and* $2m + 1$ *be two odd numbers. Then their sum is* $(2n + 1) + (2m + 1) = 2n + 2m + 2 = 2(n + m + 1)$. *It is divisible by 2, and therefore is even.*

4. Let $S = \{1, 2, 3, 4\}$ and $T = \{2, 4, 5\}$. Find the following:

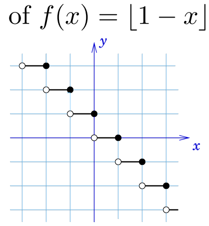
   (a) The cardinality of $S$ *is 4 (the number of elements in* $S$*)*
   (b) $S \cup T = \{1, 2, 3, 4, 5\}$ *(the union of* $S$ *and* $T$*)*
   (c) $S \cap T = \{2, 4\}$ *(the intersection of* $S$ *and* $T$*)*
   (d) $S - T = \{1, 3\}$ *(the set of elements of* $S$ *which are not in* $T$*)*
   (e) How many elements are there in $S \times T$? *12 because* $S \times T$ *consists of all pairs of the form (element of* $S$*, element of* $T$*), and there are 4 elements in* $S$ *and 3 elements in* $T$

5. Which of the following functions $\mathbb{R} \to \mathbb{R}$ are one-to-one? onto? Explain.

   (a) $f(x) = -x + 2$ *is both one-to-one and onto.*
   *One-to-one:* $f(x_1) = f(x_2) \Rightarrow -x_1 + 2 = -x_2 + 2 \Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$
   *Onto: Let* $y \in \mathbb{R}$, *then* $f(-y + 2) = -(-y + 2) + 2 = y - 2 + 2 = y$
   (b) $f(x) = e^x$ *is one-to-one but not onto.*
   *One-to-one:* $f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow \ln e^{x_1} = \ln e^{x_2} \Rightarrow x_1 = x_2$
   *Not onto because there is no* $x$ *such that* $e^x = 0$ *(since* $e^x$ *takes on only positive values)*
   (c) $f(x) = x^4$ *is neither one-to-one nor onto.*
   *Not one-to-one:* $f(1) = f(-1)$ *but* $1 \neq -1$
   *Not onto because there is no* $x$ *such that* $x^4 = -1$

6. Sketch the graphs of $f(x) = \lfloor 1 - x \rfloor$ and $g(x) = \lceil 1 - x \rceil$.

$y = \lfloor 1 - x \rfloor$




$y = \lceil -1 - x \rceil$