MATH 114
Test 3 - Solutions
December 6, 2004

1. Show that the relation $R = \{(a, b) \mid a - b \text{ is an even integer}\}$ is an equivalence relation and describe
the equivalence class of a real number $r$. What is the equivalence class of 1.5?

(i) For any $a \in \mathbb{R}$, $a - a = 0$ is an even integer, so $(a, a) \in R$. Thus $R$ is reflexive.
(ii) If $(a, b) \in R$, then $a - b$ is an even integer, $b - a = -(a - b)$ is an even integer, so $(b, a) \in R$. Thus $R$ is symmetric.
(iii) If $(a, b) \in R$ and $(b, c) \in R$ then $a - b$ and $b - c$ are even integers. Then $a - c = (a - b) + (b - c)$ is an even integer, so $(a, c) \in R$. Thus $R$ is transitive.

Therefore $R$ is an equivalence relation. The equivalence class of a number $r$ is
$[r] = \{r + 2n \mid n \in \mathbb{Z}\} = \{\ldots, r - 4, r - 2, r, r + 2, r + 4, \ldots\}$,
in particular $\{1.5\} = \{\ldots, -2.5, -0.5, 1.5, 3.5, 5.5, \ldots\}$.

2. How many solutions are there to the equation
$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$
where $x_1, x_2, x_3, x_4, x_5$ are nonnegative integers?

There is a 1-1 correspondence between solutions of this equation and arrangements of 20 stars and
4 bars where $x_1$ is the number of stars before the first bar, $x_2$ is the number of stars between the first
and the second bar, etc. The number of such arrangements (and thus the number of solutions) is
$$\binom{24}{20} = \frac{24!}{20!4!}.$$

3. (a) Draw $K_5$.

(b) How many vertices and how many edges does $K_{n,m}$ have?

$K_{n,m}$ has $n + m$ vertices ($n$ vertices in one group and $m$ vertices in the other group) and $nm$
edges (because every vertex in the first group is connected to every vertex in the second group).

4. What is the coefficient of $x^5 y^{10}$ in the expansion of $(2x - y)^{15}$?

By the Binomial Theorem, $(2x - y)^{15} = \sum_{k=0}^{15} \binom{15}{k} (2x)^{15-k} (-y)^k = \sum_{k=0}^{15} (-1)^k \binom{15}{k} 2^{15-k} x^{15-k} y^k$.

The term with $x^5 y^{10}$ corresponds to $k = 10$. Thus the coefficient is $(-1)^k \binom{15}{k} 2^{15-k}$.

5. How many strings of 8 upper case letters from the English alphabet contain exactly two $A$s and
exactly three $B$s?

There are $\binom{8}{2}$ ways to choose the two positions of $A$s. After they have been chosen there are $\binom{6}{3}$
ways to choose the three positions of $B$s. For each of the remaining 3 positions we have 24 choices
(any letter except $A$ and $B$). Therefore there are $\binom{8}{2} \binom{6}{3} 24^3$ such strings.

6. 40 different numbers are chosen from the set $\{1, 2, \ldots, 100\}$. Show that there are at least 4 different
pairs of numbers with the same sum.

There are $\binom{40}{2} = \frac{40 \cdot 39}{2} = 780$ pairs. The smallest possible sum is $1 + 2 = 3$. The largest possible
sum is $99 + 100 = 199$. Therefore there are 197 possible sums. Think of pairs as “objects” and
possible sums as “boxes”. By the generalized Dirichlet’s principle, at least \[
\left\lfloor \frac{780}{199} \right\rfloor = 4 \text{ pairs have the same sum.}
\]