MATH 114

Test 3 - Solutions

December 6, 2004

1. Show that the relation $R = \{(a,b) \mid a-b \text{ is an even integer}\}$ is an equivalence relation and describe the equivalence class of a real number r. What is the equivalence class of 1.5?

   (i) For any $a \in \mathbb{R}$, $a - a = 0$ *is an even integer, so* $(a,a) \in R$. *Thus* R *is reflexive.*
   (ii) If $(a,b) \in \mathbb{R}$, *then* $a-b$ *is an even integer,* $b - a = -(a-b)$ *is an even integer, so* $(b,a) \in \mathbb{R}$. *Thus* R *is symmetric.*
   (iii) If $(a,b) \in \mathbb{R}$ *and* $(b,c) \in \mathbb{R}$ *then* $a-b$ *and* $b-c$ *are even integers. Then* $a - c = (a-b)+(b-c)$ *is an even integer, so* $(a,c) \in \mathbb{R}$. *Thus* R *is transitive.*
   Therefore R *is an equivalence relation. The equivalence class of a number* r *is*
   $[r] = \{r + 2n \mid n \in \mathbb{Z}\} = \{\ldots, r-4, r-2, r, r+2, r+4, \ldots\}$,
   in particular $\{[1.5] = \{\ldots, -2.5, -.5, 1.5, 3.5, 5.5, \ldots\}$.

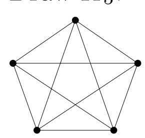
2. How many solutions are there to the equation

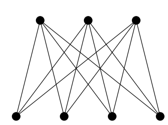
   $$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

   where x_1, x_2, x_3, x_4, x_5 are nonnegative integers?

   There is a 1-1 correspondence between solutions of this equation and arrangements of 20 stars and 4 bars where x_1 *is the number of stars before the first bar,* x_2 *is the number of stars between the first and the second bar, etc. The number of such arrangements (and thus the number of solutions) is* $\binom{24}{20} = \dfrac{24!}{20!4!}$.

3. (a) Draw K_5. (b) Draw $K_{3,4}$.

   (b) How many vertices and how many edges does $K_{n,m}$ have?

   $K_{n,m}$ *has* $n+m$ *vertices (*n *vertices in one group and* m *vertices in the other group) and* nm *edges (because every vertex in the first group is connected to every vertex in the second group).*

4. What is the coefficient of x^5y^{10} in the expansion of $(2x-y)^{15}$?

   By the Binomial Theorem, $(2x-y)^{15} = \displaystyle\sum_{k=0}^{15} \binom{15}{k} (2x)^{15-k}(-y)^k = \sum_{k=0}^{15} (-1)^k \binom{15}{k} 2^{15-k} x^{15-k} y^k$.

   The term with x^5y^{10} *corresponds to* $k = 10$. *Thus the coefficient is* $(-1)^k \binom{15}{k} 2^{15-k}$.

5. How many strings of 8 upper case letters from the English alphabet contain exactly two As and exactly three Bs?

   There are $\binom{8}{2}$ *ways to choose the two positions of* A*s. After they have been chosen there are* $\binom{6}{3}$ *ways to choose the three positions of* B*s. For each of the remaining 3 positions we have 24 choices (any letter except A and B). Therefore there are* $\binom{8}{2}\binom{6}{3}24^3$ *such strings.*

6. 40 different numbers are chosen from the set $\{1, 2, \ldots, 100\}$. Show that there are at least 4 different pairs of these numbers with the same sum.

   There are $\binom{40}{2} = \dfrac{40 \cdot 39}{2} = 780$ *pairs. The smallest possible sum is* $1 + 2 = 3$. *The largest possible sum is* $99 + 100 = 199$. *Therefore there are 197 possible sums. Think of pairs as "objects" and possible sums as "boxes". By the generalized Dirichlet's principle, at least* $\left\lceil \dfrac{780}{199} \right\rceil = 4$ *pairs have the same sum.*