uct of the sum of the divisors of s and the sum of the divisors of t.

45. Prove that the integer $2^{p-1}(2^p - 1)$ is perfect when $2^p - 1$ is a Mersenne prime.

****46.** Prove that if n is an even integer that is perfect, then $n = 2^p(2^p - 1)$, where $2^p - 1$ is a Mersenne prime.

47. Prove or disprove that if you have an eight-gallon jug of water and two empty jugs with capacities of five gallons and three gallons, respectively, then you can measure four gallons by successively pouring some of or all of the water in a jug into another jug.

***48.** Prove or disprove that if n is a positive integer, then $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor$.

49. Show that the problem of determining whether a program with a given input ever prints the digit 1 is unsolvable.

50. Show that the problem of deciding whether a specific program with a specific input halts is solvable.

51. Show that the following problem is solvable. Given two programs with their inputs and the knowledge that exactly one of them halts, determine which halts.

3.2 Sequences and Summations

INTRODUCTION

Sequences are used to represent ordered lists of elements. Sequences are used in discrete mathematics in many ways. They can be used to represent solutions to certain counting problems, as we will see in Chapter 6. They are also an important data structure in computer science. This section contains a review of the concept of a function, as well as the notation used to represent sequences and sums of terms of sequences.

When the elements of an infinite set can be listed, the set is called countable. We will conclude this section with a discussion of both countable and uncountable sets. We will prove that the set of rational numbers is countable, but the set of real numbers is not.

SEQUENCES

A sequence is a discrete structure used to represent an ordered list.

DEFINITION 1 A *sequence* is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \ldots\}$ or the set $\{1, 2, 3, \ldots\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a *term* of the sequence.

We use the notation $\{a_n\}$ to describe the sequence. (Note that a_n represents an individual term of the sequence $\{a_n\}$. Also note that the notation $\{a_n\}$ for a sequence conflicts with the notation for a set. However, the context in which we use this notation will always make it clear when we are dealing with sets and when we are dealing with sequences. Note also that the choice of the letter a is arbitrary.)

We describe sequences by listing the terms of the sequence in order of increasing subscripts.

EXAMPLE 1 Consider the sequence $\{a_n\}$, where

$$a_n = 1/n.$$

The list of the terms of this sequence, beginning with a_1, namely,

$$a_1, a_2, a_3, a_4, \ldots,$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

DEFINITION 2 A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n,$$

where the *initial term* a and the *common ratio* r are real numbers.

Remark: A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

EXAMPLE 2 The sequences $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2 \cdot 5^n$, and $\{d_n\}$ with $d_n = 6 \cdot (1/3)^n$ are geometric progressions with initial term and common ratio equal to -1 and -1; 10 and 5; and 2 and $1/3$, respectively. The list of terms $b_1, b_2, b_3, b_4, \ldots$ begins with

$$-1, 1, -1, 1, \ldots;$$

the list of terms $c_1, c_2, c_3, c_4, \ldots$ begins with

$$10, 50, 250, 1250, \ldots;$$

and the list of terms $d_1, d_2, d_3, d_4, \ldots$ begins with

$$2, 2/3, 2/9, 2/27, \ldots.$$

DEFINITION 3 An *arithmetic progression* is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd,$$

where the *initial term* a and the *common difference* d are real numbers.

Remark: An arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$.

EXAMPLE 3 The sequences $\{s_n\}$ with $s_n = -1 + 4n$ and $\{t_n\}$ with $t_n = 7 - 3n$ are both arithmetic progressions with initial terms and common differences equal to -1 and 4, and 7 and -3, respectively. The list of terms, starting with the term with $n = 0$, $s_0, s_1, s_2, s_3, \ldots$ begins with

$$-1, 3, 7, 11, \ldots,$$

and the list of terms $t_0, t_1, t_2, t_3, \ldots$ begins with

$$7, 4, 1, -2, \ldots.$$

Sequences of the form $a_1, a_2, \ldots, a_n$ are often used in computer science. These finite sequences are also called **strings.** This string is also denoted by $a_1 a_2 \cdots a_n$. (Recall that bit strings, which are finite sequences of bits, were introduced in Section 1.1.) The **length** of the string S is the number of terms in this string. The **empty string,** denoted by λ, is the string that has no terms. The empty string has length zero.

EXAMPLE 4 The string *abcd* is a string of length four.

SPECIAL INTEGER SEQUENCES

A common problem in discrete mathematics is finding a formula or a general rule for constructing the terms of a sequence. Sometimes only a few terms of a sequence solving a problem are known; the goal is to identify the sequence. Even though the initial terms of a sequence do not determine the entire sequence (after all, there are infinitely many different sequences that start with any finite set of initial terms), knowing the first few terms may help you make an educated conjecture about the identity of your sequence. Once you have made this conjecture, you can try to verify that you have the correct sequence.

When trying to deduce a possible formula or rule for the terms of a sequence from the initial terms, try to find a pattern in these terms. You might also see whether you can determine how a term might have been produced from those preceding it. There are many questions you could ask, but some of the more useful are:

- Are there runs of the same value?
- Are terms obtained from previous terms by adding the same amount or an amount that depends on the position in the sequence?
- Are terms obtained from previous terms by multiplying by a particular amount?
- Are terms obtained by combining previous terms in a certain way?
- Are there cycles among the terms?

EXAMPLE 5 Find formulas for the sequences with the following first five terms: (a) $1, 1/2, 1/4, 1/8, 1/16$ (b) $1, 3, 5, 7, 9$ (c) $1, -1, 1, -1, 1$.

Extra Examples

Solution: (a) We recognize that the denominators are powers of 2. The sequence with $a_n = 1/2^{n-1}$ is a possible match. This proposed sequence is a geometric progression with $a = 1$ and $r = 1/2$.

(b) We note that each term is obtained by adding 2 to the previous term. The sequence with $a_n = 2n - 1$ is a possible match. This proposed sequence is an arithmetic progression with $a = 1$ and $d = 2$.

(c) The terms alternate between 1 and -1. The sequence with $a_n = (-1)^{n+1}$ is a possible match. This proposed sequence is a geometric progression with $a = 1$ and $r = -1$. ◀

Examples 6 and 7 illustrate how we can analyze sequences to find how the terms are constructed.

EXAMPLE 6 How can we produce the terms of a sequence if the first 10 terms are $1, 2, 2, 3, 3, 3, 4, 4, 4, 4$?

Solution: Note that the integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, and the integer 4 appears four times. A reasonable rule for generating this sequence is that the integer n appears exactly n times, so the next five terms of the sequence would all be 5, the following six terms would all be 6, and so on. The sequence generated this way is a possible match. ◀

EXAMPLE 7 How can we produce the terms of a sequence if the first 10 terms are $5, 11, 17, 23, 29, 35, 41, 47, 53, 59$?

Solution: Note that each of the first 10 terms of this sequence after the first is obtained by adding 6 to the previous term. (We could see this by noticing that the difference between consecutive terms is 6.) Consequently, the nth term could be produced by starting with 5 and adding 6 a total of $n-1$ times; that is, a reasonable guess is that the nth term is $5+6(n-1)=6n-1$. (This is an arithmetic progression with $a=5$ and $d=6$.) ◀

Another useful technique for finding a rule for generating the terms of a sequence is to compare the terms of a sequence of interest with the terms of a well-known integer sequence, such as terms of an arithmetic progression, terms of a geometric progression, perfect squares, perfect cubes, and so on. The first 10 terms of some sequences you may want to keep in mind are displayed in Table 1.

EXAMPLE 8 Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are $1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047$.

Solution: To attack this problem, we begin by looking at the difference of consecutive terms, but we do not see a pattern. When we form the ratio of consecutive terms to see whether each term is a multiple of the previous term, we find that this ratio, although not a constant, is close to 3. So it is reasonable to suspect that the terms of this sequence are generated by a formula involving 3^n. Comparing these terms with the corresponding terms of the sequence $\{3^n\}$, we notice that the nth term is 2 less than the corresponding power of 3. We see that $a_n = 3^n - 2$ for $1 \le n \le 10$ and conjecture that this formula holds for all n. ◀

We will see throughout this text that integer sequences appear in a wide range of contexts in discrete mathematics. Sequences we have or will encounter include the sequence of prime numbers (Chapter 2), the number of ways to order n discrete objects (Chapter 4), the number of the moves required to solve the famous Tower of Hanoi puzzle with n disks (Chapter 6), and the number of rabbits on an island after n months (Chapter 6).

Links

Integer sequences appear in an amazingly wide range of subject areas besides discrete mathematics, including biology, engineering, chemistry, and physics, as well as in puzzles. A wonderfully diverse collection of over 8000 different integer sequences has been constructed over the past 20 years by the mathematician Neil Sloane, who has teamed up with Simon Plouffe, to produce *The Encyclopedia of Integer Sequences* ([SlPl95]). An extended list of the sequences is available on the Web, with new sequences added regularly. There is also a program accessible via the Web that you can use to find sequences from the encyclopedia that match initial terms you provide.

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots$
n^3	$1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, \ldots$
n^4	$1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, \ldots$
2^n	$2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots$
3^n	$3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, \ldots$
$n!$	$1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, \ldots$

SUMMATIONS

Next, we introduce **summation notation.** We begin by describing the notation used to express the sum of the terms

$$a_m, a_{m+1}, \ldots, a_n$$

from the sequence $\{a_n\}$. We use the notation

$$\sum_{j=m}^{n} a_j \qquad \text{or} \qquad \textstyle\sum_{j=m}^{n} a_j$$

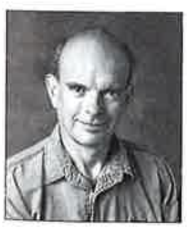
to represent

$$a_m + a_{m+1} + \cdots + a_n.$$

Here, the variable j is called the **index of summation,** and the choice of the letter j as the variable is arbitrary; that is, we could have used any other letter, such as i or k. Or, in notation,

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k.$$

Here, the index of summation runs through all integers starting with its **lower limit** m and ending with its **upper limit** n. The uppercase Greek letter sigma, Σ, is used to denote summation. We give some examples of summation notation.



Links

NEIL SLOANE (BORN 1939) Neil Sloane studied mathematics and electrical engineering at the University of Melbourne on a scholarship from the Australian state telephone company. He mastered many telephone-related jobs, such as erecting telephone poles, in his summer work. After graduating, he designed minimal cost telephone networks in Australia. In 1962 he came to the United States and studied electrical engineering at Cornell University. His Ph.D. thesis was on what are now called neural networks. He took a job at Bell Labs in 1969, working in many areas, including network design, coding theory, and sphere packing. He now works for AT&T Labs, moving there from Bell Labs when AT&T split up in 1996. One of his favorite problems is the **kissing problem** (a name he coined), which asks how many spheres can be arranged in n dimensions so that they all touch a central sphere of the same size. (In two dimensions the answer is 6, since 6 pennies can be placed so that they touch a central penny. In three dimensions, 12 billiard balls can be placed so that they touch a central billiard ball. Two billiard balls that just touch are said to "kiss," giving rise to the terminology "kissing problem" and "kissing number.") Sloane, together with Andrew Odlyzko, showed that in 8 and 24 dimensions the optimal kissing numbers are, respectively, 240 and 196,560. The kissing number is known in dimensions 1, 2, 3, 8, and 24, but not in any other dimensions. Sloane's books include *Sphere Packings, Lattices and Groups,* 3d ed., with John Conway; *The Theory of Error-Correcting Codes* with Jessie MacWilliams; *The Encyclopedia of Integer Sequences* with Simon Plouffe; and *The Rock-Climbing Guide to New Jersey Crags* with Paul Nick. The last book demonstrates his interest in rock climbing; it includes more than 50 climbing sites in New Jersey.

EXAMPLE 9 Express the sum of the first 100 terms of the sequence $\{a_n\}$, where $a_n = 1/n$ for $n = 1, 2, 3, \ldots$.

Extra Examples

Solution: The lower limit for the index of summation is 1, and the upper limit is 100. We write this sum as

$$\sum_{j=1}^{100} \frac{1}{j}.$$

◀

EXAMPLE 10 What is the value of $\sum_{j=1}^{5} j^2$?

Solution: We have

$$\begin{aligned}\sum_{j=1}^{5} j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55.\end{aligned}$$

◀

EXAMPLE 11 What is the value of $\sum_{k=4}^{8} (-1)^k$?

Solution: We have

$$\begin{aligned}\sum_{k=4}^{8} (-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8 \\ &= 1 + (-1) + 1 + (-1) + 1 \\ &= 1.\end{aligned}$$

◀

Sometimes it is useful to shift the index of summation in a sum. This is often done when two sums need to be added but their indices of summation do not match. When shifting an index of summation, it is important to make the appropriate changes in the corresponding summand. This is illustrated by the following example.

EXAMPLE 12 Suppose we have the sum

$$\sum_{j=1}^{5} j^2$$

but want the index of summation to run between 0 and 4 rather than from 1 to 5. To do this, we let $k = j - 1$. Then the new summation index runs from 0 to 4, and the term j^2 becomes $(k + 1)^2$. Hence

$$\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k + 1)^2.$$

It is easily checked that both sums are $1 + 4 + 9 + 16 + 25 = 55$. ◀

Extra Examples

Sums of terms of geometric progressions commonly arise (such sums are called **geometric series**). Theorem 1 gives us a formula for the sum of terms of a geometric progression.

THEOREM 1 If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Proof: Let

$$S = \sum_{j=0}^{n} ar^j.$$

To compute S, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$\begin{aligned} rS &= r\sum_{j=0}^{n} ar^j \\ &= \sum_{j=0}^{n} ar^{j+1} \\ &= \sum_{k=1}^{n+1} ar^k \\ &= \sum_{k=0}^{n} ar^k + (ar^{n+1} - a) \\ &= S + (ar^{n+1} - a). \end{aligned}$$

We obtain this equality by shifting the index of summation, setting $k = j + 1$.

From these equalities, we see that

$$rS = S + (ar^{n+1} - a).$$

Solving for S shows that if $r \neq 1$

$$S = \frac{ar^{n+1} - a}{r - 1}.$$

If $r = 1$, then clearly the sum equals $(n + 1)a$. ◁

EXAMPLE 13 Double summations arise in many contexts (as in the analysis of nested loops in computer programs). An example of a double summation is

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij.$$

To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:

$$\begin{aligned} \sum_{i=1}^{4}\sum_{j=1}^{3} ij &= \sum_{i=1}^{4}(i + 2i + 3i) \\ &= \sum_{i=1}^{4} 6i \\ &= 6 + 12 + 18 + 24 = 60. \end{aligned}$$

◀

We can also use summation notation to add all values of a function, or terms of an indexed set, where the index of summation runs over all values in a set. That is, we write

$$\sum_{s \in S} f(s)$$

to represent the sum of the values $f(s)$, for all members s of S.

EXAMPLE 14 What is the value of $\sum_{s \in \{0,2,4\}} s$?

Solution: Since $\sum_{s \in \{0,2,4\}} s$ represents the sum of the values of s for all the members of the set $\{0, 2, 4\}$, it follows that

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6.$$

◀

Certain sums arise repeatedly throughout discrete mathematics. Having a collection of formulae for such sums can be useful, so Table 2 provides a small table of formulae for commonly occurring sums.

We derived the first formula in this table in Theorem 1. The next three formulae give us the sum of the first n positive integers, the sum of their squares, and the sum of their cubes. These three formulae can be derived in many different ways (for example, see Exercises 21 and 22 at the end of this section). Also note that each of these formulae, once known, can easily be proved using mathematical induction, the subject of Section 3.3. The last two formulae in the table involve infinite series and will be discussed shortly.

Example 15 illustrates how the formulae in Table 2 can be useful.

EXAMPLE 15 Find $\sum_{k=50}^{100} k^2$.

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\dfrac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\dfrac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\dfrac{1}{(1-x)^2}$

Solution: First note that since $\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$, we have

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2.$$

Using the formula $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ from Table 2, we see that

$$\sum_{k=50}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338{,}350 - 40{,}425 = 297{,}925.$$ ◀

SOME INFINITE SERIES Although most of the summations in this book are finite sums, infinite series are important in some parts of discrete mathematics. The closed forms for the infinite series in Examples 16 and 17 are quite useful.

EXAMPLE 16 (Requires calculus) Let x be a real number with $|x| < 1$. Find $\sum_{n=0}^{\infty} x^n$.

Extra Examples

Solution: By Theorem 1 with $a = 1$ and $r = x$ we see that $\sum_{n=0}^{k} x^n = \frac{x^{k+1}-1}{x-1}$. Because $|x| < 1$, x^{k+1} approaches 0 as k approaches infinity. It follows that

$$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} \frac{x^{k+1}-1}{x-1} = \frac{-1}{x-1} = \frac{1}{1-x}.$$ ◀

We can produce new summation formulae by differentiating or integrating existing formulae.

EXAMPLE 17 (Requires calculus) Differentiating both sides of the equation

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x},$$

from Example 16, we find that

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}.$$

(This differentiation is valid for $|x| < 1$ by a theorem about infinite series.) ◀

CARDINALITY

Recall that in Section 1.6, the cardinality of a finite set was defined to be the number of elements in the set. It is possible to extend the concept of cardinality to all sets, both finite and infinite, with Definition 4.

DEFINITION 4 The sets A and B have the same *cardinality* if and only if there is a one-to-one correspondence from A to B.

To see that this definition agrees with the previous definition of the cardinality of a finite set as the number of elements in that set, note that there is a one-to-one correspondence between any two finite sets with n elements, where n is a nonnegative integer.

We will now split infinite sets into two groups, those with the same cardinality as the set of natural numbers and those with different cardinality.

DEFINITION 5 A set that is either finite or has the same cardinality as the set of positive integers is called *countable*. A set that is not countable is called *uncountable*.

We now give examples of countable and uncountable sets.

EXAMPLE 18 Show that the set of odd positive integers is a countable set.

Solution: To show that the set of odd positive integers is countable, we will exhibit a one-to-one correspondence between this set and the set of positive integers. Consider the function

$$f(n) = 2n - 1$$

from $\mathbf{Z}^+$ to the set of odd positive integers. We show that f is a one-to-one correspondence by showing that it is both one-to-one and onto. To see that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n - 1 = 2m - 1$, so that $n = m$. To see that it is onto, suppose that t is an odd positive integer. Then t is 1 less than an even integer $2k$, where k is a natural number. Hence $t = 2k - 1 = f(k)$. We display this one-to-one correspondence in Figure 1. ◀

Extra Examples

An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers). The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$, where $a_1 = f(1), a_2 = f(2), \ldots, a_n = f(n), \ldots$. For instance, the set of odd integers can be listed in a sequence $a_1, a_2, \ldots, a_n, \ldots$, where $a_n = 2n - 1$.

EXAMPLE 19 Show that the set of positive rational numbers is countable.

Solution: It may seem surprising that the set of positive rational numbers is countable, but we will show how we can list the positive rational numbers as a sequence $r_1, r_2, \ldots, r_n, \ldots$. First, note that every positive rational number is the quotient p/q of two positive integers. We can arrange the positive rational numbers by listing those with denominator $q = 1$ in the first row, those with denominator $q = 2$ in the second row, and so on, as displayed in Figure 2.

The key to listing the rational numbers in a sequence is to first list the positive rational numbers p/q with $p + q = 2$, followed by those with $p + q = 3$, followed by those with $p + q = 4$, and so on, following the path shown in Figure 2. Whenever we encounter a number p/q that is already listed, we do not list it again. For example, when we come to $2/2 = 1$ we do not list it since we have already listed $1/1 = 1$. The initial terms in the list of positive rational numbers we have constructed are 1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4, 5, and so on. Because all rational numbers are listed once, as the reader can verify, we have shown that the set of rational numbers is countable. ◀

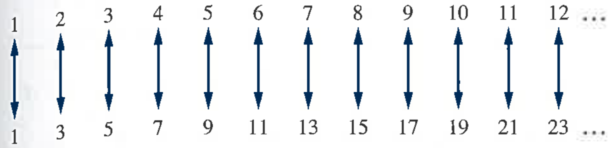


FIGURE 1 **A One-to-One Correspondence Between $\mathbf{Z}^+$ and the Set of Odd Positive Integers.**

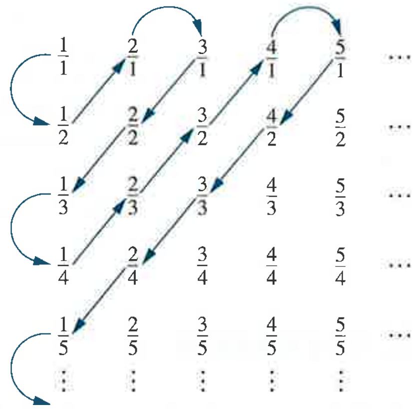


FIGURE 2 **The Positive Rational Numbers Are Countable.**

Example 20 shows that the set of real numbers is uncountable. Georg Cantor discovered this fact in 1879. We use an important proof method, known as the **Cantor diagonalization argument,** to prove that the set of real numbers is not countable. This proof method is used extensively in mathematical logic and in the theory of computation.

EXAMPLE 20 Show that the set of real numbers is an uncountable set.

Solution: To show that the set of real numbers is uncountable, we suppose that the set of real numbers is countable and arrive at a contradiction. Then, the subset of all real numbers that fall between 0 and 1 would also be countable (since any subset of a countable set is also countable; see Exercise 34 at the end of the section). Under this assumption, the real numbers between 0 and 1 can be listed in some order, say, $r_1, r_2, r_3, \ldots$. Let the decimal representation of these real numbers be

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}\ldots \\ r_4 &= 0.d_{41}d_{42}d_{43}d_{44}\ldots \\ &\ \ \vdots \end{aligned}$$

where $d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. (For example, if $r_1 = 0.23794102\ldots$, we have $d_{11} = 2$, $d_{12} = 3$, $d_{13} = 7$, and so on.) Then, form a new real number with decimal expansion $r = 0.d_1d_2d_3d_4\ldots$, where the decimal digits are determined by the following rule:

$$d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4. \end{cases}$$

(As an example, suppose that $r_1 = 0.23794102\ldots$, $r_2 = 0.44590138\ldots$, $r_3 = 0.09118764\ldots$, $r_4 = 0.80553900\ldots$, and so on. Then we have $r = 0.d_1d_2d_3d_4\ldots = 0.4544\ldots$, where $d_1 = 4$ since $d_{11} \neq 4$, $d_2 = 5$ since $d_{22} = 4$, $d_3 = 4$ since $d_{33} \neq 4$, $d_4 = 4$ since $d_{44} \neq 4$, and so on.)

Every real number has a unique decimal expansion (when the possibility that the expansion has a tail end that consists entirely of the digit 9 is excluded). Then, the real number r is not equal to any of $r_1, r_2, \ldots$, since the decimal expansion of r differs from the decimal expansion of r_i in the ith place to the right of the decimal point, for each i.

Since there is a real number r between 0 and 1 that is not in the list, the assumption that all the real numbers between 0 and 1 could be listed must be false. Therefore, all the real numbers between 0 and 1 cannot be listed, so that the set of real numbers between 0 and 1 is uncountable. Any set with an uncountable subset is uncountable (see Exercise 35 at the end of this section). Hence, the set of real numbers is uncountable. ◀

Exercises

1. Find these terms of the sequence $\{a_n\}$ where $a_n = 2 \cdot (-3)^n + 5^n$.

a) a_0 **b)** a_1 **c)** a_4 **d)** a_5

2. What is the term a_8 of the sequence $\{a_n\}$ if a_n equals

a) 2^{n-1}? **b)** 7?
c) $1 + (-1)^n$? **d)** $-(-2)^n$?

3. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals

a) $2^n + 1$? **b)** $(n + 1)^{n+1}$?
c) $\lfloor n/2 \rfloor$? **d)** $\lfloor n/2 \rfloor + \lceil n/2 \rceil$?

4. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals

a) $(-2)^n$? **b)** 3?
c) $7 + 4^n$? **d)** $2^n + (-2)^n$?

5. List the first 10 terms of each of these sequences.

a) the sequence that begins with 2 and in which each successive term is 3 more than the preceding term
b) the sequence that lists each positive integer three times, in increasing order
c) the sequence that lists the odd positive integers in increasing order, listing each odd integer twice
d) the sequence whose nth term is $n! - 2^n$
e) the sequence that begins with 3, where each succeeding term is twice the preceding term
f) the sequence whose first two terms are 1 and each succeeding term is the sum of the two preceding terms (This is the famous Fibonacci sequence, which we will study later in this text.)
g) the sequence whose nth term is the number of bits in the binary expansion of the number n (defined in Section 2.5)
h) the sequence where the nth term is the number of letters in the English word for the index n

6. List the first 10 terms of each of these sequences.

a) the sequence obtained by starting with 10 and obtaining each term by subtracting 3 from the previous term
b) the sequence whose nth term is the sum of the first n positive integers
c) the sequence whose nth term is $3^n - 2^n$
d) the sequence whose nth term is $\lfloor \sqrt{n} \rfloor$
e) the sequence whose first two terms are 1 and 2 and each succeeding term is the sum of the two previous terms
f) the sequence whose nth term is the largest integer whose binary expansion (defined in Section 2.5) has n bits (Write your answer in decimal notation.)
g) the sequence whose terms are constructed sequentially as follows: start with 1, then add 1, then multiply by 1, then add 2, then multiply by 2, and so on
h) the sequence whose nth term is the largest integer k such that $k! \leq n$

7. Find at least three different sequences beginning with the terms 1, 2, 4 whose terms are generated by a simple formula or rule.

8. Find at least three different sequences beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule.

9. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list.

a) $1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, \ldots$
b) $1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, \ldots$
c) $1, 0, 2, 0, 4, 0, 8, 0, 16, 0, \ldots$
d) $3, 6, 12, 24, 48, 96, 192, \ldots$
e) $15, 8, 1, -6, -13, -20, -27, \ldots$
f) $3, 5, 8, 12, 17, 23, 30, 38, 47, \ldots$
g) $2, 16, 54, 128, 250, 432, 686, \ldots$
h) $2, 3, 7, 25, 121, 721, 5041, 40321, \ldots$

10. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list.

a) $3, 6, 11, 18, 27, 38, 51, 66, 83, 102, \ldots$
b) $7, 11, 15, 19, 23, 27, 31, 35, 39, 43, \ldots$
c) $1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, \ldots$
d) $1, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, \ldots$
e) $0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, \ldots$

f) 1, 3, 15, 105, 945, 10395, 135135, 2027025, 34459425, ...
g) 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, ...
h) 2, 4, 16, 256, 65536, 4294967296, ...

***11.** Show that if a_n denotes the nth positive integer that is not a perfect square, then $a_n = n + \{\sqrt{n}\}$, where $\{x\}$ denotes the integer closest to the real number x.

***12.** Let a_n be the nth term of the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, ..., constructed by including the integer k exactly k times. Show that $a_n = \lfloor \sqrt{2n} + \frac{1}{2} \rfloor$.

13. What are the values of these sums?

a) $\sum_{k=1}^{5} (k+1)$ **b)** $\sum_{j=0}^{4} (-2)^j$

c) $\sum_{i=1}^{10} 3$ **d)** $\sum_{j=0}^{8} (2^{j+1} - 2^j)$

14. What are the values of these sums, where $S = \{1, 3, 5, 7\}$?

a) $\sum_{j \in S} j$ **b)** $\sum_{j \in S} j^2$

c) $\sum_{j \in S} (1/j)$ **d)** $\sum_{j \in S} 1$

15. What is the value of each of these sums of terms of a geometric progression?

a) $\sum_{j=0}^{8} 3 \cdot 2^j$ **b)** $\sum_{j=1}^{8} 2^j$

c) $\sum_{j=2}^{8} (-3)^j$ **d)** $\sum_{j=0}^{8} 2 \cdot (-3)^j$

16. Find the value of each of these sums.

a) $\sum_{j=0}^{8} (1 + (-1)^j)$ **b)** $\sum_{j=0}^{8} (3^j - 2^j)$

c) $\sum_{j=0}^{8} (2 \cdot 3^j + 3 \cdot 2^j)$ **d)** $\sum_{j=0}^{8} (2^{j+1} - 2^j)$

17. Compute each of these double sums.

a) $\sum_{i=1}^{2} \sum_{j=1}^{3} (i + j)$ **b)** $\sum_{i=0}^{2} \sum_{j=0}^{3} (2i + 3j)$

c) $\sum_{i=1}^{3} \sum_{j=0}^{2} i$ **d)** $\sum_{i=0}^{2} \sum_{j=1}^{3} ij$

18. Compute each of these double sums.

a) $\sum_{i=1}^{3} \sum_{j=1}^{2} (i - j)$ **b)** $\sum_{i=0}^{3} \sum_{j=0}^{2} (3i + 2j)$

c) $\sum_{i=1}^{3} \sum_{j=0}^{2} j$ **d)** $\sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3$

19. Show that $\sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - a_0$ where $a_0, a_1, \ldots, a_n$ is a sequence of real numbers. This type of sum is called **telescoping.**

20. Use the identity $1/(k(k+1)) = 1/k - 1/(k+1)$ and Exercise 19 to compute $\sum_{k=1}^{n} 1/(k(k+1))$.

21. Sum both sides of the identity $k^2 - (k-1)^2 = 2k - 1$ from $k = 1$ to $k = n$ and use Exercise 19 to find

a) a formula for $\sum_{k=1}^{n} (2k - 1)$ (the sum of the first n odd natural numbers).
b) a formula for $\sum_{k=1}^{n} k$.

***22.** Use the technique given in Exercise 19, together with the result of Exercise 21b, to find a formula for $\sum_{k=1}^{n} k^2$.

23. Find $\sum_{k=100}^{200} k$. (Use Table 2.)

24. Find $\sum_{k=99}^{200} k^3$. (Use Table 2.)

***25.** Find a formula for $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$, when m is a positive integer. (*Hint:* Use the formula for $\sum_{k=1}^{n} k^2$.)

***26.** Find a formula for $\sum_{k=0}^{m} \lfloor \sqrt[3]{k} \rfloor$, when m is a positive integer. (*Hint:* Use the formula for $\sum_{k=1}^{n} k^3$.)

There is also a special notation for products. The product of $a_m, a_{m+1}, \ldots, a_n$ is represented by

$$\prod_{j=m}^{n} a_j.$$

27. What are the values of the following products?

a) $\prod_{i=0}^{10} i$ **b)** $\prod_{i=5}^{8} i$
c) $\prod_{i=1}^{100} (-1)^i$ **d)** $\prod_{i=1}^{10} 2$

Recall that the value of the factorial function at a positive integer n, denoted by $n!$, is the product of the positive integers from 1 to n, inclusive. Also, we specify that $0! = 1$.

28. Express $n!$ using product notation.

29. Find $\sum_{j=0}^{4} j!$.

30. Find $\prod_{j=0}^{4} j!$.

31. Determine whether each of these sets is countable or uncountable. For those that are countable, exhibit a one-to-one correspondence between the set of natural numbers and that set.

a) the negative integers
b) the even integers
c) the real numbers between 0 and $\frac{1}{2}$
d) integers that are multiples of 7

***32.** Determine whether each of these sets is countable or uncountable. For those that are countable, exhibit a one-to-one correspondence between the set of natural numbers and that set.

a) integers not divisible by 3
b) integers divisible by 5 but not by 7
c) the real numbers with decimal representations consisting of all 1s
d) the real numbers with decimal representations of all 1s or 9s

33. If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?

34. Show that a subset of a countable set is also countable.

35. Show that if A is an uncountable set and $A \subseteq B$, then B is uncountable.

36. Show that the union of two countable sets is countable.

**37. Show that the union of a countable number of countable sets is countable.

38. Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable.

*39. Show that the set of all bit strings is countable.

*40. Show that the set of real numbers that are solutions of quadratic equations $ax^2 + bx + c = 0$, where a, b, and c are integers, is countable.

*41. Show that the set of all computer programs in a particular programming language is countable. (*Hint:* A computer program written in a programming language can be thought of as a string of symbols from a finite alphabet.)

*42. Show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable. [*Hint:* First set up a one-to-one correspondence between the set of real numbers between 0 and 1 and a subset of these functions. Do this by associating to the real number $0.d_1d_2 \ldots d_n \ldots$ the function f with $f(n) = d_n$.]

*43. We say that a function is **computable** if there is a computer program that finds the values of this function. Use Exercises 41 and 42 to show that there are functions that are not computable.

*44. Prove that the set of positive rational numbers is countable by setting up a function that assigns to a rational number p/q with $\gcd(p, q) = 1$ the base 11 number formed from the decimal representation of p followed by the base 11 digit A, which corresponds to the decimal number 10, followed by the decimal representation of q.

*45. Prove that the set of positive rational numbers is countable by showing that the function K is a one-to-one correspondence between the set of positive rational numbers and the set of positive integers if $K(m/n) = p_1^{2a_1} p_2^{2a_2} \cdots p_s^{2a_s} q_1^{2b_1-1} q_2^{2b_2-1} \cdots q_t^{2b_t-1}$, where $\gcd(m, n) = 1$ and the prime-power factorizations of m and n are $m = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$ and $n = q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t}$.

3.3 Mathematical Induction

INTRODUCTION

Links

What is a formula for the sum of the first n positive odd integers? The sums of the first n positive odd integers for $n = 1, 2, 3, 4, 5$ are

$$1 = 1, \qquad 1 + 3 = 4, \qquad 1 + 3 + 5 = 9,$$
$$1 + 3 + 5 + 7 = 16, \qquad 1 + 3 + 5 + 7 + 9 = 25.$$

From these values it is reasonable to guess that the sum of the first n positive odd integers is n^2. We need a method to *prove* that this *guess* is correct, if in fact it is.

Mathematical induction is an extremely important proof technique that can be used to prove assertions of this type. As we will see in this section and in subsequent chapters, mathematical induction is used extensively to prove results about a large variety of discrete objects. For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities.

In this section we will describe how mathematical induction can be used and why it is a valid proof technique. It is extremely important to note that mathematical induction can be used only to prove results obtained in some other way. It is *not* a tool for discovering formulae or theorems.

There are several useful illustrations of mathematical induction that can help you remember how this principle works. One of these involves a line of people, person one, person two, and so on. A secret is told to person one, and each person tells the secret to the next person in line, if the former person hears it. Let $P(n)$ be the proposition that person n knows the secret. Then $P(1)$ is true, since the secret is told to person one; $P(2)$ is true, since person one tells person two the secret; $P(3)$ is true, since person two tells person three the secret; and so on. By the principle of mathematical induction, every person in line learns the secret. This is illustrated in Figure 1. (Of course, it has been assumed that each person relays the secret in an unchanged manner to the next person, which is usually not true in real life.)