**37. Show that the union of a countable number of countable sets is countable.

38. Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable.

*39. Show that the set of all bit strings is countable.

*40. Show that the set of real numbers that are solutions of quadratic equations $ax^2 + bx + c = 0$, where a, b, and c are integers, is countable.

*41. Show that the set of all computer programs in a particular programming language is countable. (*Hint:* A computer program written in a programming language can be thought of as a string of symbols from a finite alphabet.)

*42. Show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable. [*Hint:* First set up a one-to-one correspondence between the set of real numbers between 0 and 1 and a subset of these functions. Do this by associating to the real number $0.d_1d_2 \ldots d_n \ldots$ the function f with $f(n) = d_n$.]

*43. We say that a function is **computable** if there is a computer program that finds the values of this function. Use Exercises 41 and 42 to show that there are functions that are not computable.

*44. Prove that the set of positive rational numbers is countable by setting up a function that assigns to a rational number p/q with $\gcd(p, q) = 1$ the base 11 number formed from the decimal representation of p followed by the base 11 digit A, which corresponds to the decimal number 10, followed by the decimal representation of q.

*45. Prove that the set of positive rational numbers is countable by showing that the function K is a one-to-one correspondence between the set of positive rational numbers and the set of positive integers if $K(m/n) = p_1^{2a_1} p_2^{2a_2} \cdots p_s^{2a_s} q_1^{2b_1-1} q_2^{2b_2-1} \cdots q_t^{2b_t-1}$, where $\gcd(m, n) = 1$ and the prime-power factorizations of m and n are $m = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$ and $n = q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t}$.

3.3 Mathematical Induction

INTRODUCTION

What is a formula for the sum of the first n positive odd integers? The sums of the first n positive odd integers for $n = 1, 2, 3, 4, 5$ are

$$1 = 1, \quad 1 + 3 = 4, \quad 1 + 3 + 5 = 9,$$
$$1 + 3 + 5 + 7 = 16, \quad 1 + 3 + 5 + 7 + 9 = 25.$$

From these values it is reasonable to guess that the sum of the first n positive odd integers is n^2. We need a method to *prove* that this *guess* is correct, if in fact it is.

Mathematical induction is an extremely important proof technique that can be used to prove assertions of this type. As we will see in this section and in subsequent chapters, mathematical induction is used extensively to prove results about a large variety of discrete objects. For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities.

In this section we will describe how mathematical induction can be used and why it is a valid proof technique. It is extremely important to note that mathematical induction can be used only to prove results obtained in some other way. It is *not* a tool for discovering formulae or theorems.

There are several useful illustrations of mathematical induction that can help you remember how this principle works. One of these involves a line of people, person one, person two, and so on. A secret is told to person one, and each person tells the secret to the next person in line, if the former person hears it. Let $P(n)$ be the proposition that person n knows the secret. Then $P(1)$ is true, since the secret is told to person one; $P(2)$ is true, since person one tells person two the secret; $P(3)$ is true, since person two tells person three the secret; and so on. By the principle of mathematical induction, every person in line learns the secret. This is illustrated in Figure 1. (Of course, it has been assumed that each person relays the secret in an unchanged manner to the next person, which is usually not true in real life.)

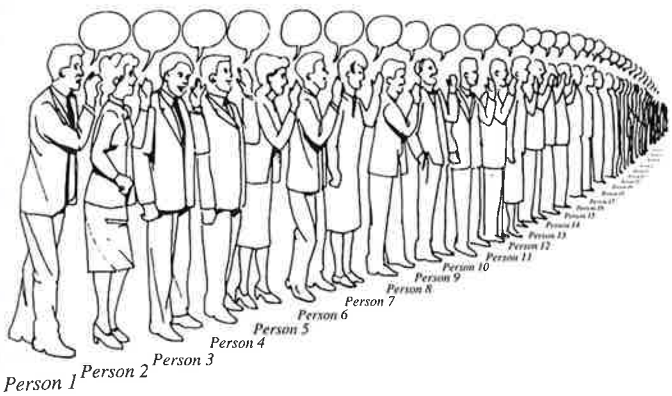

FIGURE 1 **People Telling Secrets.**

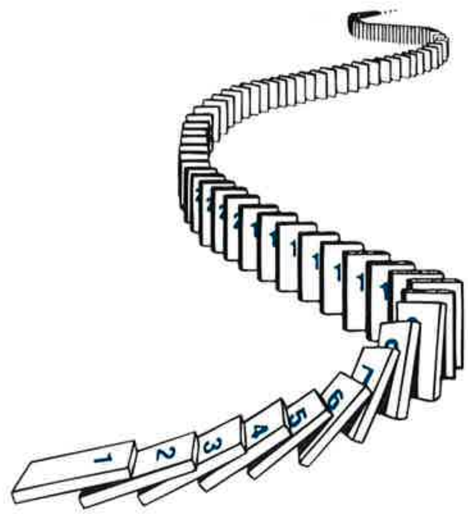

FIGURE 2 **Illustrating How Mathematical Induction Works Using Dominos.**

Another way to illustrate the principle of mathematical induction is to consider an infinite row of dominos, labeled $1, 2, 3, \ldots, n$, where each domino is standing up. Let $P(n)$ be the proposition that domino n is knocked over. If the first domino is knocked over—i.e., if $P(1)$ is true—and if, whenever the nth domino is knocked over, it also knocks the $(n+1)$th domino over—i.e., if $P(n) \to P(n+1)$ is true—then all the dominos are knocked over. This is illustrated in Figure 2.

MATHEMATICAL INDUCTION

Many theorems state that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, such as the statement that $1 + 2 + \cdots + n = n(n+1)/2$ or the statement that $n \leq 2^n$. Mathematical induction is a technique for proving theorems of this kind. In other words, mathematical induction is used to prove propositions of the form $\forall n\, P(n)$, where the universe of discourse is the set of positive integers.

A proof by mathematical induction that $P(n)$ is true for every positive integer n consists of two steps:

BASIS STEP: The proposition $P(1)$ is shown to be true.

INDUCTIVE STEP: The implication $P(k) \to P(k+1)$ is shown to be true for every positive integer k.

Here, the statement $P(k)$ for a fixed positive integer k is called the **inductive hypothesis.** When we complete both steps of a proof by mathematical induction, we have proved that $P(n)$ is true for all positive integers n; that is, we have shown that $\forall n P(n)$ is true.

Expressed as a rule of inference, this proof technique can be stated as

$$[P(1) \wedge \forall k (P(k) \to P(k+1))] \to \forall n P(n).$$

Since mathematical induction is such an important technique, it is worthwhile to explain in detail the steps of a proof using this technique. The first thing we do to prove that $P(n)$

is true for all positive integers n is to show that $P(1)$ is true. This amounts to showing that the particular statement obtained when n is replaced by 1 in $P(n)$ is true. Then we must show that $P(k) \to P(k+1)$ is true for every positive integer k. To prove that this implication is true for every positive integer k, we need to show that $P(k+1)$ cannot be false when $P(k)$ is true. This can be accomplished by assuming that $P(k)$ is true and showing that *under this hypothesis* $P(k+1)$ must also be true.

Remark: In a proof by mathematical induction it is *not* assumed that $P(k)$ is true for all positive integers! It is only shown that *if it is assumed* that $P(k)$ is true, then $P(k+1)$ is also true. Thus, a proof by mathematical induction is not a case of begging the question, or circular reasoning.

When we use mathematical induction to prove a theorem, we first show that $P(1)$ is true. Then we know that $P(2)$ is true, since $P(1)$ implies $P(2)$. Further, we know that $P(3)$ is true, since $P(2)$ implies $P(3)$. Continuing along these lines, we see that $P(n)$ is true, for every positive integer n.

EXAMPLES OF PROOFS BY MATHEMATICAL INDUCTION

Links

Extra Examples

We will use a variety of examples to illustrate how theorems are proved using mathematical induction. We begin by proving a formula for the sum of the first n odd positive integers. (Many theorems proved in this section via mathematical induction can be proved using different methods. However, it is worthwhile to try to prove a theorem in more than one way, since one method of attack may succeed whereas another approach may not.)

EXAMPLE 1 Use mathematical induction to prove that the sum of the first n odd positive integers is n^2.

Solution: Let $P(n)$ denote the proposition that the sum of the first n odd positive integers is n^2. We must first complete the basis step; that is, we must show that $P(1)$ is true. Then we must carry out the inductive step; that is, we must show that $P(k+1)$ is true when $P(k)$ is assumed to be true.

BASIS STEP: $P(1)$ states that the sum of the first one odd positive integer is 1^2. This is true since the sum of the first odd positive integer is 1.

INDUCTIVE STEP: To complete the inductive step we must show that the proposition $P(k) \to P(k+1)$ is true for every positive integer k. To do this, suppose that $P(k)$ is true for a positive integer k; that is,

$$1 + 3 + 5 + \cdots + (2k-1) = k^2.$$

Links

HISTORICAL NOTE The first known use of mathematical induction is in the work of the sixteenth-century mathematician Francesco Maurolico (1494–1575). Maurolico wrote extensively on the works of classical mathematics and made many contributions to geometry and optics. In his book *Arithmeticorum Libri Duo,* Maurolico presented a variety of properties of the integers together with proofs of these properties. To prove some of these properties he devised the method of mathematical induction. His first use of mathematical induction in this book was to prove that the sum of the first n odd positive integers equals n^2.

[Note that the kth odd positive integer is $(2k-1)$, since this integer is obtained by adding 2 a total of $k-1$ times to 1.] We must show that $P(k+1)$ is true, assuming that $P(k)$ is true. Note that $P(k+1)$ is the statement that

$$1+3+5+\cdots+(2k-1)+(2k+1)=(k+1)^2.$$

So, assuming that $P(k)$ is true, it follows that

$$\begin{aligned}1+3+5+\cdots+(2k-1)+(2k+1) &= [1+3+\cdots+(2k-1)]+(2k+1)\\ &= k^2+(2k+1)\\ &= k^2+2k+1\\ &= (k+1)^2.\end{aligned}$$

This shows that $P(k+1)$ follows from $P(k)$. Note that we used the inductive hypothesis $P(k)$ in the second equality to replace the sum of the first k odd positive integers by k^2.

Since $P(1)$ is true and the implication $P(k) \to P(k+1)$ is true for all positive integers k, the principle of mathematical induction shows that $P(n)$ is true for all positive integers n. ◀

Example 2 uses the principle of mathematical induction to prove an inequality.

EXAMPLE 2 Use mathematical induction to prove the inequality

$$n < 2^n$$

for all positive integers n.

Solution: Let $P(n)$ be the proposition "$n < 2^n$."

BASIS STEP: $P(1)$ is true, since $1 < 2^1 = 2$.

INDUCTIVE STEP: Assume that $P(k)$ is true for the positive integer k. That is, assume that $k < 2^k$. We need to show that $P(k+1)$ is true. That is, we need to show that $k+1 < 2^{k+1}$. Adding 1 to both sides of $k < 2^k$, and then noting that $1 \le 2^k$, gives

$$k+1 < 2^k+1 \le 2^k+2^k = 2^{k+1}.$$

We have shown that $P(k+1)$ is true, namely, that $k+1 < 2^{k+1}$, based on the assumption that $P(k)$ is true. The induction step is complete.

Therefore, by the principle of mathematical induction, it has been shown that $n < 2^n$ is true for all positive integers n. ◀

We will now use mathematical induction to prove a theorem involving the divisibility of integers.

EXAMPLE 3 Use mathematical induction to prove that n^3-n is divisible by 3 whenever n is a positive integer.

Solution: To construct the proof, let $P(n)$ denote the proposition: "n^3-n is divisible by 3."

BASIS STEP: $P(1)$ is true, since $1^3-1=0$ is divisible by 3.

INDUCTIVE STEP: Assume that $P(k)$ is true; that is, $k^3 - k$ is divisible by 3. We must show that $P(k+1)$ is true. That is, we must show that $(k+1)^3 - (k+1)$ is divisible by 3. Note that

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= (k^3 - k) + 3(k^2 + k).\end{aligned}$$

Since both terms in this sum are divisible by 3 (the first by the assumption of the inductive step, and the second because it is 3 times an integer), it follows that $(k+1)^3 - (k+1)$ is also divisible by 3. This completes the induction step. Thus, by the principle of mathematical induction, $n^3 - n$ is divisible by 3 whenever n is a positive integer. ◀

Sometimes we need to show that $P(n)$ is true for $n = b, b+1, b+2, \ldots,$ where b is an integer other than 1. We can use mathematical induction to accomplish this as long as we change the basis step. For instance, consider Example 4, which proves that a summation formula is valid for all nonnegative integers, so that we need to prove that $P(n)$ is true for $n = 0, 1, 2, \ldots$.

EXAMPLE 4 Use mathematical induction to show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

Solution: Let $P(n)$ be the proposition that this formula is correct for the integer n.

BASIS STEP: $P(0)$ is true since $2^0 = 1 = 2^1 - 1$.

INDUCTIVE STEP: Assume that $P(k)$ is true. To carry out the inductive step using this assumption, it must be shown that $P(k+1)$ is true, namely,

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1.$$

Using the inductive hypothesis $P(k)$, it follows that

$$\begin{aligned}1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1.\end{aligned}$$

This finishes the inductive step, which completes the proof. ◀

As Example 4 demonstrates, to use mathematical induction to show that $P(n)$ is true for $n = b, b+1, b+2, \ldots,$ where b is an integer other than 1, we show that $P(b)$ is true (the basis step) and then show that the implication $P(k) \to P(k+1)$ is true for $k = b, b+1, b+2, \ldots$ (the inductive step). Note that b can be negative, zero, or positive. Following the domino analogy we used earlier, imagine that we begin by knocking down the bth domino (the basis step), and as each domino falls, it knocks down the next domino (the inductive step). We leave it to the reader to show that this form of induction is valid (see Exercise 76).

The formula given in Example 4 is a special case of a general result for the sum of terms of a geometric progression (Theorem 1 in Section 3.2). We will use mathematical induction to provide an alternate proof of this formula.

EXAMPLE 5 **Sums of Geometric Progressions** Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}, \qquad \text{when } r \neq 1.$$

Solution: To prove this formula using mathematical induction, let $P(n)$ be the proposition that the sum of the first $n + 1$ terms of a geometric progression in this formula is correct.

BASIS STEP: $P(0)$ is true, since

$$a = \frac{ar - a}{r - 1}.$$

INDUCTIVE STEP: Assume that $P(k)$ is true. That is, assume

$$a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1} - a}{r - 1}.$$

To show that this implies that $P(k + 1)$ is true, add ar^{k+1} to both sides of this equation to obtain

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+1} - a}{r - 1} + ar^{k+1}.$$

Rewriting the right-hand side of this equation shows that

$$\frac{ar^{k+1} - a}{r - 1} + ar^{k+1} = \frac{ar^{k+1} - a}{r - 1} + \frac{ar^{k+2} - ar^{k+1}}{r - 1}$$
$$= \frac{ar^{k+2} - a}{r - 1}.$$

Combining these last two equations gives

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r - 1}.$$

This shows that if $P(k)$ is true, then $P(k + 1)$ must also be true. This completes the inductive argument and shows that the formula for the sum of the terms of a geometric series is correct. ◀

As previously mentioned, the formula in Example 4 is the case of the formula in Example 5 with $a = 1$ and $r = 2$. The reader should verify that putting these values for a and r in the general formula gives the same formula as in Example 4.

An important inequality for the sum of the reciprocals of a set of positive integers will be proved in the next example.

EXAMPLE 6 **An Inequality for Harmonic Numbers** The **harmonic numbers** H_j, $j = 1, 2, 3, \ldots$, are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}.$$

For instance,

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2},$$

whenever n is a nonnegative integer.

Solution: To carry out the proof, let $P(n)$ be the proposition that $H_{2^n} \geq 1 + n/2$.

BASIS STEP: $P(0)$ is true, since $H_{2^0} = H_1 = 1 \geq 1 + 0/2$.

INDUCTIVE STEP: Assume that $P(k)$ is true, so that $H_{2^k} \geq 1 + k/2$. It must be shown that $P(k+1)$, which states that $H_{2^{k+1}} \geq 1 + (k+1)/2$, must also be true under this assumption. This can be done since

$$\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{definition of harmonic number}\\
&= H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{definition of harmonic number}\\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the inductive hypothesis}\\
&\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} && \text{since there are } 2^k \text{ terms each not less than } 1/2^{k+1}\\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2}\\
&= 1 + \frac{k+1}{2}.
\end{aligned}$$

This establishes the inductive step of the proof. Thus, the inequality for the harmonic numbers is valid for all nonnegative integers n. ◀

Remark: The inequality established here shows that the **harmonic series**

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is a divergent infinite series. This is an important example in the study of infinite series.

Example 7 shows how mathematical induction can be used to verify a formula for the number of subsets of a finite set.

EXAMPLE 7 **The Number of Subsets of a Finite Set** Use mathematical induction to show that if S is a finite set with n elements, then S has 2^n subsets. (We will prove this result directly in several ways in Chapter 4.)

Solution: Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.

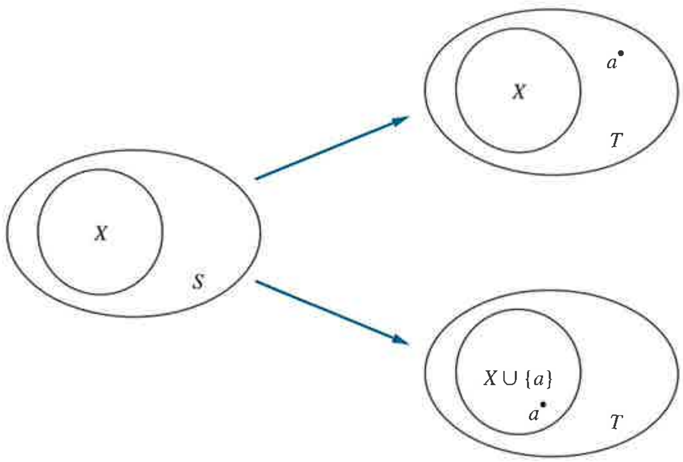

FIGURE 3 **Generating Subsets of a Set with $k+1$ Elements. Here $T = S \cup \{a\}$.**

BASIS STEP: $P(0)$ is true, since a set with zero elements, the empty set, has exactly $2^0 = 1$ subsets, since it has one subset, namely, itself.

INDUCTIVE STEP: Assume that $P(k)$ is true, that is, that every set with k elements has 2^k subsets. It must be shown that under this assumption $P(k+1)$, which is the statement that every set with $k+1$ elements has 2^{k+1} subsets, must also be true. To show this, let T be a set with $k+1$ elements. Then, it is possible to write $T = S \cup \{a\}$ where a is one of the elements of T and $S = T - \{a\}$. The subsets of T can be obtained in the following way. For each subset X of S there are exactly two subsets of T, namely, X and $X \cup \{a\}$. (This is illustrated in Figure 3.) These constitute all the subsets of T and are all distinct. Since there are 2^k subsets of S, there are $2 \cdot 2^k = 2^{k+1}$ subsets of T. This finishes the induction argument. ◀

EXAMPLE 8 Show that if n is a positive integer,

$$1 + 2 + \cdots + n = n(n+1)/2.$$

Solution: Let $P(n)$ be the proposition that the sum of the first n positive integers is $n(n+1)/2$. We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots$. Namely, we must show that $P(1)$ is true and that the implication $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, \ldots$.

BASIS STEP: $P(1)$ is true, since $1 = 1(1+1)/2$.

INDUCTIVE STEP: Assume that $P(k)$ holds so that

$$1 + 2 + \cdots + k = k(k+1)/2.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1 + 2 + \cdots + k + (k+1) = (k+1)[(k+1)+1]/2 = (k+1)(k+2)/2$$

is also true. Add $k + 1$ to both sides of the equation in $P(k)$ to obtain

$$\begin{aligned} 1 + 2 + \cdots + k + (k + 1) &= k(k + 1)/2 + (k + 1) \\ &= [(k/2) + 1](k + 1) \\ &= (k + 1)(k + 2)/2. \end{aligned}$$

This last equation shows that $P(k + 1)$ is true. This completes the inductive step and completes the proof. ◀

EXAMPLE 9 Use mathematical induction to prove that $2^n < n!$ for every positive integer n with $n \geq 4$.

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

BASIS STEP: To prove the inequality for $n \geq 4$ requires that the basis step be $P(4)$. Note that $P(4)$ is true, since $2^4 = 16 < 4! = 24$.

INDUCTIVE STEP: Assume that $P(k)$ is true. That is, assume that $2^k < k!$. We must show that $P(k + 1)$ is true. That is, we must show that $2^{k+1} < (k + 1)!$. Multiplying both sides of the inequality $2^k < k!$ by 2, it follows that

$$\begin{aligned} 2 \cdot 2^k &< 2 \cdot k! \\ &< (k + 1) \cdot k! \\ &= (k + 1)!. \end{aligned}$$

This shows that $P(k + 1)$ is true when $P(k)$ is true. This completes the inductive step of the proof. Hence, it follows that $2^n < n!$ is true for all integers n with $n \geq 4$. ◀

EXAMPLE 10 Use mathematical induction to prove the following generalization of one of De Morgan's laws:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j},$$

whenever $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and $n \geq 2$.

Solution: Let $P(n)$ be the identity for n sets.

BASIS STEP: The statement $P(2)$ asserts that $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$. This is one of De Morgan's laws; it was proved in Section 1.7.

INDUCTIVE STEP: Assume that $P(k)$ is true, that is,

$$\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_k$ are subsets of the universal set U. To carry out the inductive step it must be shown that if this equality holds for any k subsets of U, it must also be valid for any $k+1$ subsets of U. Suppose that $A_1, A_2, \ldots, A_k, A_{k+1}$ are subsets of U. When the inductive hypothesis is assumed to hold, it follows that

$$
\begin{aligned}
\overline{\bigcap_{j=1}^{k+1} A_j} &= \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}} \\
&= \overline{\left(\bigcap_{j=1}^{k} A_j\right)} \cup \overline{A_{k+1}} && \text{by De Morgan's law} \\
&= \left(\bigcup_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}} && \text{by the inductive hypothesis} \\
&= \bigcup_{j=1}^{k+1} \overline{A_j}.
\end{aligned}
$$

This completes the proof by induction. ◀

Links

Example 11 illustrates how mathematical induction can be used to prove a result about covering chessboards with pieces shaped like the letter "L."

EXAMPLE 11 Let n be a positive integer. Show that any $2^n \times 2^n$ chessboard with one square removed can be tiled using L-shaped pieces, where these pieces cover three squares at a time, as shown in Figure 4.

FIGURE 4 An L-Shaped Piece.

Solution: Let $P(n)$ be the proposition that any $2^n \times 2^n$ chessboard with one square removed can be tiled using L-shaped pieces. We can use mathematical induction to prove that $P(n)$ is true for all positive integers n.

BASIS STEP: $P(1)$ is true, since any of the four 2×2 chessboards with one square removed can be tiled using one L-shaped piece, as shown in Figure 5.

INDUCTIVE STEP: Assume that $P(k)$ is true; that is, assume that any $2^k \times 2^k$ chessboard with one square removed can be tiled using L-shaped pieces. It must be shown that under this assumption $P(k+1)$ must also be true; that is, any $2^{k+1} \times 2^{k+1}$ chessboard with one square removed can be tiled using L-shaped pieces.

To see this, consider a $2^{k+1} \times 2^{k+1}$ chessboard with one square removed. Split this chessboard into four chessboards of size $2^k \times 2^k$, by dividing it in half in both directions. This is illustrated in Figure 6. No square has been removed from three of these four

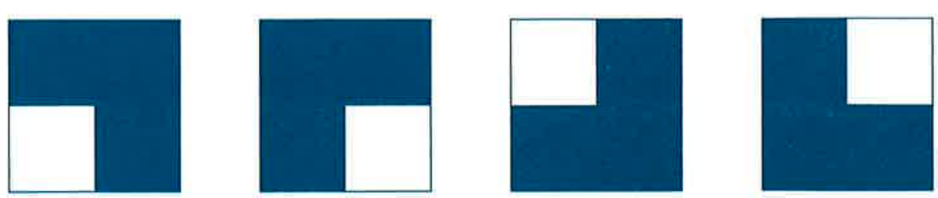

FIGURE 5 Tiling 2 × 2 Chessboards with One Square Removed.

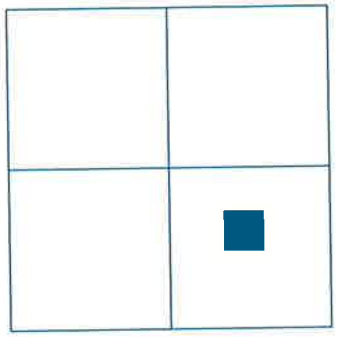

FIGURE 6 Dividing a $2^{k+1} \times 2^{k+1}$ Chessboard into Four $2^k \times 2^k$ Chessboards.

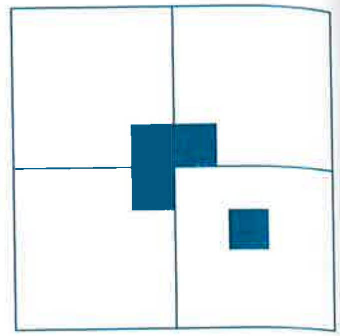

FIGURE 7 Tiling the $2^{k+1} \times 2^{k+1}$ Chessboard with One Square Removed.

chessboards. The fourth $2^k \times 2^k$ chessboard has one square removed, so by the inductive hypothesis, it can be covered by L-shaped pieces. Now temporarily remove the square from each of the other three $2^k \times 2^k$ chessboards that has the center of the original, larger chessboard as one of its corners, as shown in Figure 7. By the inductive hypothesis, each of these three $2^k \times 2^k$ chessboards with a square removed can be tiled by L-shaped pieces. Furthermore, the three squares that were temporarily removed can be covered by one L-shaped piece. Hence, the entire $2^{k+1} \times 2^{k+1}$ chessboard can be tiled with L-shaped pieces. This completes the proof. ◀

Next, we provide an example that illustrates one of many ways mathematical induction is used in the study of algorithms. We will show how mathematical induction can be used to prove that a greedy algorithm yields an optimal solution. (For an introduction to greedy algorithms, see Section 2.1.)

EXAMPLE 12 We can use a greedy algorithm to schedule a subset of m proposed talks $t_1, t_2, \ldots, t_m$ in a single lecture hall. Suppose that talk t_j begins at time b_j and ends at time e_j. (No two lectures can proceed at the same time and a lecture can begin at the same time one ends.) We assume that the talks are listed in order of nondecreasing ending time, so that $e_1 \leq e_2 \leq \cdots \leq e_m$. The greedy algorithm proceeds by selecting at each stage a talk with the earliest ending time among all those talks that begin after all talks already scheduled end. (A lecture with an earliest end time is always added first by the algorithm.) We will show that this greedy algorithm is optimal in the sense that it always schedules the most talks possible. To prove the optimality of this algorithm we use mathematical induction on the variable n, the number of talks scheduled by the algorithm. We let $P(n)$ be the proposition that if the greedy algorithm schedules n talks, then it is not possible to schedule more than n talks.

BASIS STEP: Suppose that the greedy algorithm managed to schedule just one talk, t_1. This means that every other talk cannot start after e_1, the ending time of t_1. Otherwise, the first such talk we come to as we go through the talks in order of nondecreasing end time could be added. Hence, at time e_1 each of the remaining talks needs to use the lecture hall since they all start at or before e_1 and end after e_1. It follows that no two talks can be scheduled since both need to use the lecture hall at time e_1. This shows that $P(1)$ is true and completes the basis step.

INDUCTIVE STEP: Assume that $P(k)$ is true, that is, that the greedy algorithm always schedules the most possible talks when it selects k talks, given any set of talks (no matter how large). Now assume that the algorithm has selected $k+1$ talks. We must show that the greedy algorithm has selected the largest number of talks possible, given the assumption that it always produces an optimal solution when it schedules k talks. That is, we need to show that $P(k+1)$ is true, assuming that $P(k)$ is true.

To complete the inductive step, we first show there is a schedule including the most talks possible that contains talk t_1, a talk with the earliest end time. This is easy to see since a schedule that begins with the talk t_i in the list, where $i > 1$, can be changed so that talk t_1 replaces talk t_i. To see this, note that since $e_1 \leq e_i$, all talks that were scheduled to follow talk t_i can still be scheduled.

Once we included talk t_1, scheduling the talks so that as many as possible are scheduled is reduced to scheduling as many talks as possible that begin at or after time e_1. So, if we have scheduled as many talks as possible, the schedule of talks other than talk t_1 is an optimal schedule of the original talks that begin once talk t_1 has ended. Since the greedy algorithm schedules k talks when it creates this schedule, we can apply the induction hypothesis to conclude that it has scheduled the most possible talks. It follows that the greedy algorithm has scheduled the most possible talks, $k+1$, when it produced a schedule with $k+1$ talks, so that $P(k+1)$ is true. This completes the induction step, finishing the proof that $P(n)$ is true for all positive integers n, and completes the proof of optimality. ◀

STRONG INDUCTION

There is another form of mathematical induction that is often useful in proofs. With this form we use the same basis step as before, but we use a different inductive step. We assume that $P(j)$ is true for $j = 1, \ldots, k$ and show that $P(k+1)$ must also be true based on this assumption. This is called **strong induction** (and is sometimes also known as the **second principle of mathematical induction**).

We summarize the two steps used to show that $P(n)$ is true for all positive integers n:

BASIS STEP: The proposition $P(1)$ is shown to be true.

INDUCTIVE STEP: It is shown that $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for every positive integer k.

The two forms of mathematical induction are equivalent; that is, each can be shown to be a valid proof technique assuming the other. We leave it as an exercise for the reader to show this. We now give three examples that show how the strong induction is used.

EXAMPLE 13 Consider a game in which two players take turns removing any number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially, the second player can always guarantee a win.

Solution: Let n be the number of matches in each pile. We will use strong induction to prove $P(n)$, the statement that the second player can win when there are initially n matches in each pile.

BASIS STEP: When $n = 1$, the first player has only one choice, removing one match from one of the piles, leaving a single pile with a single match, which the second player can remove to win the game.

INDUCTIVE STEP: Suppose that $P(j)$ is true for all j with $1 \le j \le k$, that is, that the second player can always win whenever there are j matches where $1 \le j \le k$ in each of the two piles at the start of the game. Now suppose that there are $k+1$ matches in each of the two piles at the start of the game and suppose that the first player removes j matches $(1 \le j \le k)$ from one of the piles, leaving $k+1-j$ matches in this pile. By removing the same number of matches from the other pile, player two creates the situation where there are two piles each with $k+1-j$ matches. Because $1 \le k+1-j \le k$ the second player can always win by the induction hypothesis. We complete the proof by noting that if the first player removes all $k+1$ matches from one of the piles, the second player can win by removing all the remaining matches. ◀

EXAMPLE 14 Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution: Let $P(n)$ be the proposition that n can be written as the product of primes.

BASIS STEP: $P(2)$ is true, since 2 can be written as the product of one prime, itself. [Note that $P(2)$ is the first case we need to establish.]

INDUCTIVE STEP: Assume that $P(j)$ is true for all positive integers j with $j \le k$. To complete the inductive step, it must be shown that $P(k+1)$ is true under this assumption.

There are two cases to consider, namely, when $k+1$ is prime and when $k+1$ is composite. If $k+1$ is prime, we immediately see that $P(k+1)$ is true. Otherwise, $k+1$ is composite and can be written as the product of two positive integers a and b with $2 \le a \le b < k+1$. By the induction hypothesis, both a and b can be written as the product of primes. Thus, if $k+1$ is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b. ◀

Remark: Since 1 is a product of primes, namely, the *empty* product of no primes, we could have started the proof in Example 14 with $P(1)$ as the basis step. We chose not to do this because many people find this confusing.

Note that Example 14 completes the proof of the Fundamental Theorem of Arithmetic, which asserts that every nonnegative integer can be written uniquely as the product of primes in nondecreasing order. We showed in Section 2.6 (see page 183) that an integer has at most one such factorization into primes. Example 14 shows there is at least one such factorization.

Using the principle of mathematical induction, instead of strong induction, to prove the result in Example 14 is difficult. However, as Example 15 shows, some results can be readily proved using either the principle of mathematical induction or strong induction.

EXAMPLE 15 Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: We will prove this result using the principle of mathematical induction. Then we will present a proof using strong induction. Let $P(n)$ be the statement that postage of n cents can be formed using 4-cent and 5-cent stamps.

We begin by using the principle of mathematical induction.

BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.

INDUCTIVE STEP: Assume that $P(k)$ is true, so that postage of k cents can be formed using 4-cent and 5-cent stamps. If at least one 4-cent stamp was used, replace it with a 5-cent stamp to form postage of $k+1$ cents. If no 4-cent stamps were used, postage of k cents was formed using just 5-cent stamps. Since $k \geq 12$, at least three 5-cent stamps were used. So, replace three 5-cent stamps with four 4-cent stamps to form postage of $k + 1$ cents. This completes the inductive step, as well as the proof by the principle of mathematical induction.

Next, we will use strong induction. We will show that postage of 12, 13, 14, and 15 cents can be formed and then show how to get postage of $k + 1$ cents for $k \geq 15$ from postage of $k - 3$ cents.

BASIS STEP: We can form postage of 12, 13, 14, and 15 cents using three 4-cent stamps, two 4-cent stamps and one 5-cent stamp, one 4-cent stamp and two 5-cent stamps, and three 5-cent stamps, respectively.

INDUCTIVE STEP: Let $k \geq 15$. Assume that we can form postage of j cents, where $12 \leq j \leq k$. To form postage of $k + 1$ cents, use the stamps that form postage of $k - 3$ cents together with a 4-cent stamp. This completes the inductive step, as well as the proof by strong induction.

(There are other ways to approach this problem besides those described here. Can you find a solution that does not use mathematical induction?) ◀

Remark: Example 15 shows how we can adapt strong induction to handle cases where the inductive step is valid only for sufficiently large values of k. In particular, to prove that $P(n)$ is true for $n = j, j + 1, j + 2, \ldots,$ where j is an integer, we first show that $P(j), P(j + 1), P(j + 2), \ldots, P(l)$ are true (the basis step), and then we show that $[P(j) \wedge P(j + 1) \wedge P(j + 2) \wedge \cdots \wedge P(k)] \rightarrow P(k + 1)$ is true for every integer $k \geq l$ (the inductive step). For example, the basis step of the second proof in the solution of Example 15 shows that $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are true. We need to prove these cases separately since the inductive step, which shows that $[P(12) \wedge P(13) \wedge \cdots \wedge P(k)] \rightarrow P(k + 1)$, holds only when $k \geq 15$.

THE WELL-ORDERING PROPERTY

The validity of mathematical induction follows from the following fundamental axiom about the set of integers.

THE WELL-ORDERING PROPERTY Every nonempty set of nonnegative integers has a least element.

The well-ordering property can often be used directly in proofs.

EXAMPLE 16 Use the well-ordering property to prove the division algorithm. Recall that the division algorithm states that if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$ and $a = dq + r$.

Solution: Let S be the set of nonnegative integers of the form $a - dq$ where q is an integer. This set is nonempty since $-dq$ can be made as large as desired (taking q to be a negative integer with large absolute value). By the well-ordering property S has a least element $r = a - dq_0$.

The integer r is nonnegative. It is also the case that $r < d$. If it were not, then there would be a smaller nonnegative element in S, namely, $a - d(q_0 + 1)$. To see this, suppose that $r \geq d$. Since $a = dq_0 + r$, it follows that $a - d(q_0 + 1) = (a - dq_0) - d = r - d \geq 0$.

Consequently, there are integers q and r with $0 \leq r < d$. The proof that q and r are unique is left as an exercise for the reader. ◀

EXAMPLE 17 In a round-robin tournament every player plays every other player exactly once and each match has a winner and loser. We say that the players $p_1, p_2, \ldots, p_m$ form a *cycle* if p_1 beats p_2, p_2 beats $p_3, \ldots, p_{m-1}$ beats p_m, and p_m beats p_1. Use the well-ordering principle to show that if there is a cycle of length m ($m \geq 3$) among the players in a round-robin tournament, there must be a cycle of three of these players.

Solution: We assume that there is no cycle of three players. Since there is at least one cycle in the round-robin tournament, the set of all positive integers n for which there is a cycle of length n is nonempty. By the well-ordering property, this set of positive integers has a least element k, which by assumption must be greater than three. Consequently, there exists a cycle of players $p_1, p_2, p_3, \ldots, p_k$ and no shorter cycle exists.

Now suppose that there is no cycle of three of these players, so that $k > 3$. Consider the first three elements of this cycle, p_1, p_2, p_3. There are two possible outcomes of the match between p_1 and p_3. If p_3 beats p_1, it follows that p_1, p_2, p_3 is a cycle of length three, contradicting our assumption that there is no cycle of three players. Consequently, it must be the case that p_1 beats p_3. This means that we can omit p_2 from the cycle $p_1, p_2, p_3, \ldots, p_k$ to obtain the cycle $p_1, p_3, p_4, \ldots, p_k$ of length $k - 1$, contradicting the assumption that the smallest cycle has length k. We conclude that there must be a cycle of length three. ◀

INFINITE DESCENT We will now describe a proof method, the **method of infinite descent,** introduced by Pierre de Fermat in the 1600s. The method of infinite descent is often used to show that for a propositional function $P(n)$, $P(k)$ is false for all positive integers k. The method is based on the observation that if $P(k)$ is true for at least one integer k, then the well-ordering property implies that there is a least positive integer s such that $P(s)$ is true. The method proceeds by finding a positive integer s' with $s' < s$ for which $P(s')$ is true. It follows that $P(n)$ must be false for all positive integers. (This technique is called the method of *infinite* descent since the procedure of finding smaller integers for which the propositional function is true could be continued indefinitely, producing an infinite sequence of decreasing positive integers, which is impossible by the well-ordering property.) The method of infinite descent is often used to show that there are no solutions in integers to certain equations. In particular, Fermat used it to prove the $n = 4$ case of Fermat's Last Theorem, which states that the equation $x^4 + y^4 = z^4$ has no solutions in positive integers. We illustrate the use of infinite descent in Example 18.

EXAMPLE 18 In Example 21 in Section 1.5 we showed that $\sqrt{2}$ is irrational. Here we will provide a different proof of this fact using infinite descent. First, suppose that $\sqrt{2}$ is rational. Then there exist positive integers m and n such that $\sqrt{2} = m/n$. By the well-ordering property, there is a least positive integer N such that $\sqrt{2} = M/N$ for some positive integer M. (This would make N the smallest possible denominator of ratios of two positive integers that equal $\sqrt{2}$.)

To carry out the proof by infinite descent, we will show that $\sqrt{2} = (2N-M)/(M-N)$ and $0 < M - N < N$. This contradicts the choice of N as the least positive integer such that $\sqrt{2} = M/N$ for some positive integer M. To show that $\sqrt{2} = (2N - M)/(M - N)$ we need only show that $(2N - M)/(M - N) = M/N$. To show this, first note that because $(M/N)^2 = 2$, it follows that $M^2 = 2N^2$. Consequently,

$$\frac{2N - M}{M - N} = \frac{(2N - M)N}{(M - N)N} = \frac{2N^2 - MN}{(M - N)N} = \frac{M^2 - MN}{(M - N)N} = \frac{(M - N)M}{(M - N)N} = \frac{M}{N}.$$

To finish the proof, we need only show that the denominator, $M - N$, is positive and smaller than N. To see this, note that because $1 < \sqrt{2} < 2$ and $\sqrt{2} = M/N$, it follows that $1 < M/N < 2$, and hence, that $N < M < 2N$. Subtracting N, we conclude that $0 < M - N < N$. ◀

WHY MATHEMATICAL INDUCTION IS VALID

Why is mathematical induction a valid proof technique? The reason comes from the well-ordering property. Suppose we know that $P(1)$ is true and that the proposition $P(k) \to P(k+1)$ is true for all positive integers k. To show that $P(n)$ must be true for all positive integers, assume that there is at least one positive integer for which $P(n)$ is false. Then the set S of positive integers for which $P(n)$ is false is nonempty. Thus, by the well-ordering property, S has a least element, which will be denoted by m. We know that m cannot be 1, since $P(1)$ is true. Since m is positive and greater than 1, $m - 1$ is a positive integer. Furthermore, since $m - 1$ is less than m, it is not in S, so $P(m-1)$ must be true. Since the implication $P(m-1) \to P(m)$ is also true, it must be the case that $P(m)$ is true. This contradicts the choice of m. Hence, $P(n)$ must be true for every positive integer n.

Exercises

1. Find a formula for the sum of the first n even positive integers.

2. Use mathematical induction to prove the formula that you found in Exercise 1.

3. Use mathematical induction to prove that $3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^n = 3(5^{n+1} - 1)/4$ whenever n is a nonnegative integer.

4. Use mathematical induction to prove that $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = (1 - (-7)^{n+1})/4$ whenever n is a nonnegative integer.

5. Find a formula for

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

by examining the values of this expression for small values of n. Use mathematical induction to prove your result.

6. Find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n. Use mathematical induction to prove your result.

7. Show that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ whenever n is a positive integer.

8. Show that $1^3 + 2^3 + \cdots + n^3 = [n(n+1)/2]^2$ whenever n is a positive integer.

9. Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever n is a nonnegative integer.

10. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

***11.** Show by mathematical induction that if $h > -1$, then $1 + nh \le (1+h)^n$ for all nonnegative integers n. This is called **Bernoulli's inequality.**

12. Prove that $3^n < n!$ whenever n is a positive integer greater than 6.

13. Show that $2^n > n^2$ whenever n is an integer greater than 4.

14. Use mathematical induction to prove that $n! < n^n$ whenever n is a positive integer greater than 1.

15. Prove using mathematical induction that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3$$

whenever n is a positive integer.

16. Use mathematical induction to prove that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4.$$

17. Show that $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}n(n+1)/2$ whenever n is a positive integer.

18. Prove that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

whenever n is a positive integer greater than 1.

19. Show that any postage that is a positive integer number of cents greater than 7 cents can be formed using just 3-cent stamps and 5-cent stamps.

20. Use mathematical induction to show that 3 divides $n^3 + 2n$ whenever n is a nonnegative integer.

21. Use mathematical induction to show that 5 divides $n^5 - n$ whenever n is a nonnegative integer.

22. Use mathematical induction to show that 6 divides $n^3 - n$ whenever n is a nonnegative integer.

***23.** Use mathematical induction to show that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

24. Use mathematical induction to show that $n^2 - 7n + 12$ is nonnegative if n is an integer greater than 3.

25. Use mathematical induction to prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

***26.** Use mathematical induction to prove that a set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly three elements whenever n is an integer greater than or equal to 3.

27. Use mathematical induction to prove that $\sum_{j=1}^{n} j^4 = n(n+1)(2n+1)(3n^2+3n-1)/30$ whenever n is a positive integer.

28. For which nonnegative integers n is $n^2 \leq n!$? Prove your answer using mathematical induction.

29. For which nonnegative integers n is $2n + 3 \leq 2^n$? Prove your answer using mathematical induction.

30. Use mathematical induction to show that $1/(2n) \leq [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)]/(2 \cdot 4 \cdot \cdots \cdot 2n)$ whenever n is a positive integer.

31. **a)** Determine which amounts of postage can be formed using just 5-cent and 6-cent stamps.
b) Prove your answer to (a) using the principle of mathematical induction.
c) Prove your answer to (a) using the second principle of mathematical induction.

32. Which amounts of money can be formed using just dimes and quarters? Prove your answer using a form of mathematical induction.

33. An automatic teller machine has only \$20 bills and \$50 bills. Which amounts of money can the machine dispense, assuming the machine has a limitless supply of these two denominations of bills? Prove your answer using a form of mathematical induction.

34. Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The bar or a smaller rectangular piece of the bar can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

35. Consider this variation of the game of Nim. The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction to show that if each player plays the best strategy possible, the first player wins if $n = 4j, 4j+2$, or $4j+3$ for some nonnegative integer j and the second player wins in the remaining case when $n = 4j+1$ for some nonnegative integer j.

36. Prove that $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$ using mathematical induction.

37. Show that if n is a positive integer, then

$$\sum_{\{a_1,\ldots,a_k\}\subseteq\{1,2,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} = n.$$

(Here the sum is over all nonempty subsets of the set of the n smallest positive integers.)

38. Use mathematical induction to show that given a set of $n+1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set.

***39.** A knight on a chessboard can move one space horizontally (in either direction) and two spaces vertically (in either direction) or two spaces horizontally (in either direction) and one space vertically (in either direction). Use mathematical induction to show that for every square a knight starting at $(0, 0)$, the corner of an infinite chessboard made up of all squares (m, n), where m and n are nonnegative integers, can visit this square using a finite sequence of moves. (*Hint:* Use induction on the variable $s = m + n$.)

40. Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n-1)/2$.

41. (Calculus required) Use mathematical induction to prove that the derivative of $f(x) = x^n$ equals nx^{n-1} whenever n is a positive integer. (For the inductive step, use the product rule for derivatives.)

42. Suppose that

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

where a and b are real numbers. Show that

$$\mathbf{A}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

for every positive integer n.

43. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are square matrices with the property $\mathbf{AB} = \mathbf{BA}$. Show that $\mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$ for every positive integer n.

44. Suppose that m is a positive integer. Use mathematical induction to prove that if a and b are integers with $a \equiv b \pmod{m}$, then $a^k \equiv b^k \pmod{m}$ whenever k is a nonnegative integer.

45. Use mathematical induction to show that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B).$$

46. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_k \subseteq B_k$ for $k = 1, 2, \ldots, n$, then

a) $\bigcup_{k=1}^{n} A_k \subseteq \bigcup_{k=1}^{n} B_k$. **b)** $\bigcap_{k=1}^{n} A_k \subseteq \bigcap_{k=1}^{n} B_k$.

47. Use mathematical induction to prove that if $A_1, A_2, \ldots, A_n$ are subsets of a universal set U, then

$$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}.$$

48. Use mathematical induction to show that $\neg(p_1 \vee p_2 \vee \cdots \vee p_n)$ is equivalent to $\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$ whenever $p_1, p_2, \ldots, p_n$ are propositions.

***49.** Show that

$$[(p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \cdots \wedge (p_{n-1} \to p_n)] \to [(p_1 \wedge p_2 \wedge \cdots \wedge p_{n-1}) \to p_n]$$

is a tautology whenever $p_1, p_2, \ldots, p_n$ are propositions.

50. What is wrong with this "proof"?

"Theorem" For every positive integer n, $\sum_{i=1}^{n} i = (n + \frac{1}{2})^2/2$.

Basis Step: The formula is true for $n = 1$.

Inductive Step: Suppose that $\sum_{i=1}^{n} i = (n + \frac{1}{2})^2/2$. Then $\sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) + (n + 1)$. By the inductive hypothesis, $\sum_{i=1}^{n+1} i = (n + \frac{1}{2})^2/2 + n + 1 = (n^2 + n + \frac{1}{4})/2 + n + 1 = (n^2 + 3n + \frac{9}{4})/2 = (n + \frac{3}{2})^2/2 = [(n + 1) + \frac{1}{2}]^2/2$, completing the inductive step.

51. What is wrong with this "proof" that all horses are the same color?

Let $P(n)$ be the proposition that all the horses in a set of n horses are the same color.

Basis Step: Clearly, $P(1)$ is true.

Inductive Step: Assume that $P(k)$ is true, so that all the horses in any set of k horses are the same color. Consider any $k + 1$ horses; number these as horses $1, 2, 3, \ldots, k, k + 1$. Now the first k of these horses all must have the same color, and the last k of these must also have the same color. Since the set of the first k horses and the set of the last k horses overlap, all $k + 1$ must be the same color. This shows that $P(k + 1)$ is true and finishes the proof by induction.

52. What is wrong with this "proof"?

"Theorem" For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.

Basis Step: Suppose that $n = 1$. If $\max(x, y) = 1$ and x and y are positive integers, we have $x = 1$ and $y = 1$.

Inductive Step: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x and y are positive integers, then $x = y$. Now let $\max(x, y) = k + 1$, where x and y are positive integers. Then $\max(x - 1, y - 1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

53. What is wrong with this "proof" by strong induction?

"Theorem" For every nonnegative integer n, $5n = 0$.

Basis Step: $5 \cdot 0 = 0$.

Inductive Step: Suppose that $5j = 0$ for all nonnegative integers j with $0 \le j \le k$. Write $k + 1 = i + j$, where i and j are natural numbers less than $k + 1$. By the induction hypothesis, $5(k + 1) = 5(i + j) = 5i + 5j = 0 + 0 = 0$.

***54.** Find the flaw with the following "proof" that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

Basis Step: $a^0 = 1$ is true by the definition of a^0.

Inductive Step: Assume that $a^j = 1$ for all nonnegative integers j with $j \le k$. Then note that

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

***55.** Show that strong induction is a valid method of proof by showing that it follows from the well-ordering property.

***56.** Show that the following form of mathematical induction is a valid method to prove that $P(n)$ is true for all positive integers n.

Basis Step: $P(1)$ and $P(2)$ are true.

Inductive Step: For each positive integer k, if $P(k)$ and $P(k + 1)$ are both true, then $P(k + 2)$ is true.

In Exercises 57 and 58, H_n denotes the nth harmonic number.

***57.** Use mathematical induction to show that $H_{2^n} \le 1 + n$ whenever n is a nonnegative integer.

***58.** Use mathematical induction to prove that

$$H_1 + H_2 + \cdots + H_n = (n + 1)H_n - n.$$

***59.** Prove that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n + 1} - 1).$$

***60.** Show that n lines separate the plane into $(n^2 + n + 2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

61. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. The **arithmetic mean of these numbers is defined by

$$A = (a_1 + a_2 + \cdots + a_n)/n,$$

and the **geometric mean** of these numbers is defined by

$$G = (a_1 a_2 \cdots a_n)^{1/n}.$$

Use mathematical induction to prove that $A \geq G$.

*62. Use mathematical induction to show that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer.

63. Use mathematical induction to prove Lemma 2 of Section 2.6, which states that if p is a prime and $p \mid a_1 a_2 \cdots a_n$, where a_i is an integer for $i = 1, 2, 3, \ldots, n$, then $p \mid a_i$ for some integer i.

64. Use infinite descent to show that the equation $8x^4 + 4y^4 + 2z^4 = w^4$ has no solutions in positive integers x, y, z, and w.

65. Use infinite descent to show that there are no solutions in positive integers w, x, y, and z to $w^2 + x^2 + y^2 + z^2 = 2wxyz$. (*Hint:* First show that if this equation holds, then all of w, x, y, and z must be even. Then show that all four of these integers must be divisible by 4, by 8, and so on.)

*66. The well-ordering property can be used to show that there is a unique greatest common divisor of two positive integers. Let a and b be positive integers, and let S be the set of positive integers of the form $as + bt$, where s and t are integers.

a) Show that S is nonempty.
b) Use the well-ordering property to show that S has a smallest element c.
c) Show that if d is a common divisor of a and b, then d is a divisor of c.
d) Show that $c \mid a$ and $c \mid b$. (*Hint:* First, assume that $c \nmid a$. Then $a = qc + r$, where $0 < r < c$. Show that $r \in S$, contradicting the choice of c.)
e) Conclude from (c) and (d) that the greatest common divisor of a and b exists. Finish the proof by showing that this greatest common divisor of two positive integers is unique.

*67. Show that if $a_1, a_2, \ldots, a_n$ are n distinct real numbers, exactly $n - 1$ multiplications are used to compute the product of these n numbers no matter how parentheses are inserted into their product. (*Hint:* Use strong induction and consider the last multiplication.)

68. Construct a tiling using L-shaped pieces of the 4×4 chessboard with the square in the upper left corner removed.

69. Construct a tiling using L-shaped pieces of the 8×8 chessboard with the square in the upper left corner removed.

70. Prove or disprove that all chessboards of these shapes can be completely covered using L-shaped pieces whenever n is a positive integer.

a) 3×2^n b) 6×2^n
c) $3^n \times 3^n$ d) $6^n \times 6^n$

*71. Show that a three-dimensional $2^n \times 2^n \times 2^n$ chessboard with one $1 \times 1 \times 1$ cube missing can be completely covered by $2 \times 2 \times 2$ cubes with one $1 \times 1 \times 1$ cube removed.

*72. Show that an $n \times n$ chessboard with one square removed can be completely covered using L-shaped pieces if $n > 5$, n is odd, and $3 \nmid n$.

73. Show that a 5×5 chessboard with a corner square removed can be tiled using L-shaped pieces.

*74. Find a 5×5 chessboard with a square removed that cannot be tiled using L-shaped pieces. Prove that such a tiling does not exist for this board.

75. Let a be an integer and d be a positive integer. Show that the integers q and r with $a = dq + r$ and $0 \leq r < d$, which were shown to exist in Example 16, are unique.

76. Use the principle of mathematical induction to show that $P(n)$ is true for $n = b, b + 1, b + 2, \ldots$, where b is an integer, if $P(b)$ is true and the implication $P(k) \rightarrow P(k + 1)$ is true for all positive integers k with $k \geq b$.

**77. Can you use the well-ordering property to prove this statement? "Every positive integer can be described using no more than 15 English words"?

78. Use the well-ordering principle to show that if x and y are real numbers with $x < y$, then there is a rational number r with $x < r < y$. [*Hint:* Show that there exists a positive integer A with $A > 1/(y - x)$. Then show that there is a rational number r with denominator A between x and y by looking at the numbers $\lfloor x \rfloor + j/A$, where j is a positive integer.]

3.4 Recursive Definitions and Structural Induction

INTRODUCTION

Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in terms of itself. This process is called **recursion.** For instance, the picture shown in Figure 1 is produced recursively. First, an original picture is given. Then a process of