DEFINITION 10 Let $\mathbf{A}$ be a square zero–one matrix and let r be a positive integer. The rth *Boolean power* of $\mathbf{A}$ is the Boolean product of r factors of $\mathbf{A}$. The rth Boolean product of $\mathbf{A}$ is denoted by $\mathbf{A}^{[r]}$. Hence

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \mathbf{A} \odot \cdots \odot \mathbf{A}}_{r \text{ times}}.$$

(This is well defined because the Boolean product of matrices is associative.) We also define $\mathbf{A}^{[0]}$ to be $\mathbf{I}_n$.

EXAMPLE 9 Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find $\mathbf{A}^{[n]}$ for all positive integers n.

Solution: We find that

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We also find that

$$\mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Additional computation shows that

$$\mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The reader can now see that $\mathbf{A}^{[n]} = \mathbf{A}^{[5]}$ for all positive integers n with $n \geq 5$. ◀

Exercises

1. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 4 & 6 \\ 1 & 1 & 3 & 7 \end{bmatrix}$.

a) What size is $\mathbf{A}$?
b) What is the third column of $\mathbf{A}$?
c) What is the second row of $\mathbf{A}$?
d) What is the element of $\mathbf{A}$ in the (3, 2)th position?
e) What is $\mathbf{A}^t$?

2. Find $\mathbf{A} + \mathbf{B}$, where

a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix}$,

$\mathbf{B} = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}$.

b) $\mathbf{A} = \begin{bmatrix} -1 & 0 & 5 & 6 \\ -4 & -3 & 5 & -2 \end{bmatrix}$,

$\mathbf{B} = \begin{bmatrix} -3 & 9 & -3 & 4 \\ 0 & -2 & -1 & 2 \end{bmatrix}$.

3. Find $\mathbf{AB}$ if

a) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$.

b) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$.

c) $\mathbf{A} = \begin{bmatrix} 4 & -3 \\ 3 & -1 \\ 0 & -2 \\ -1 & 5 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 3 & 2 & -2 \\ 0 & -1 & 4 & -3 \end{bmatrix}$.

4. Find the product **AB**, where

a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

b) $\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ 1 & 2 & 2 \\ 2 & 1 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 0 & 3 & -1 \\ -3 & -2 & 0 & 2 \end{bmatrix}$.

c) $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 7 & 2 \\ -4 & -3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 & 2 & 3 & 0 \\ -2 & 0 & 3 & 4 & 1 \end{bmatrix}$.

5. Find a matrix **A** such that

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

[*Hint:* Finding **A** requires that you solve systems of linear equations.]

6. Find a matrix **A** such that

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 4 & 0 & 3 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 7 & 1 & 3 \\ 1 & 0 & 3 \\ -1 & -3 & 7 \end{bmatrix}.$$

7. Let **A** be an $m \times n$ matrix and let **0** be the $m \times n$ matrix that has all entries equal to zero. Show that $\mathbf{A} = \mathbf{0} + \mathbf{A} = \mathbf{A} + \mathbf{0}$.

8. Show that matrix addition is commutative; that is, show that if **A** and **B** are both $m \times n$ matrices, then $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

9. Show that matrix addition is associative; that is, show that if **A**, **B**, and **C** are all $m \times n$ matrices, then $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.

10. Let **A** be a 3×4 matrix, **B** be a 4×5 matrix, and **C** be a 4×4 matrix. Determine which of the following products are defined and find the size of those that are defined.

a) **AB** **b)** **BA** **c)** **AC**
d) **CA** **e)** **BC** **f)** **CB**

11. What do we know about the sizes of the matrices **A** and **B** if both of the products **AB** and **BA** are defined?

12. In this exercise we show that matrix multiplication is distributive over matrix addition.

a) Suppose that **A** and **B** are $m \times k$ matrices and that **C** is a $k \times n$ matrix. Show that $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.

b) Suppose that **C** is an $m \times k$ matrix and that **A** and **B** are $k \times n$ matrices. Show that $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$.

13. In this exercise we show that matrix multiplication is associative. Suppose that **A** is an $m \times p$ matrix, **B** is a $p \times k$ matrix, and **C** is a $k \times n$ matrix. Show that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

14. The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is called a **diagonal matrix** if $a_{ij} = 0$ when $i \neq j$. Show that the product of two $n \times n$ diagonal matrices is again a diagonal matrix. Give a simple rule for determining this product.

15. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find a formula for $\mathbf{A}^n$, whenever n is a positive integer.

16. Show that $(\mathbf{A}^t)^t = \mathbf{A}$.

17. Let **A** and **B** be two $n \times n$ matrices. Show that

a) $(\mathbf{A} + \mathbf{B})^t = \mathbf{A}^t + \mathbf{B}^t$.

b) $(\mathbf{AB})^t = \mathbf{B}^t\mathbf{A}^t$.

If **A** and **B** are $n \times n$ matrices with $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$, then **B** is called the **inverse** of **A** (this terminology is appropriate because such a matrix **B** is unique) and **A** is said to be **invertible**. The notation $\mathbf{B} = \mathbf{A}^{-1}$ denotes that **B** is the inverse of **A**.

18. Show that

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

is the inverse of

$$\begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}.$$

19. Let **A** be the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that if $ad - bc \neq 0$, then

$$\mathbf{A}^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}.$$

20. Let

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}.$$

a) Find $\mathbf{A}^{-1}$. [*Hint:* Use Exercise 19.]

b) Find $\mathbf{A}^3$.

c) Find $(\mathbf{A}^{-1})^3$.

d) Use your answers to (b) and (c) to show that $(\mathbf{A}^{-1})^3$ is the inverse of $\mathbf{A}^3$.

21. Let **A** be an invertible matrix. Show that $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$ whenever n is a positive integer.

22. Let **A** be a matrix. Show that the matrix $\mathbf{AA}^t$ is symmetric. [*Hint:* Show that this matrix equals its transpose with the help of Exercise 17b.]

23. Suppose that **A** is an $n \times n$ matrix where n is a positive integer. Show that $\mathbf{A} + \mathbf{A}^t$ is symmetric.

24. a) Show that the system of simultaneous linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n. \end{aligned}$$

in the variables $x_1, x_2, \ldots, x_n$ can be expressed as $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A} = [a_{ij}]$, $\mathbf{X}$ is an $n \times 1$ matrix with x_i the entry in its ith row, and $\mathbf{B}$ is an $n \times 1$ matrix with b_i the entry in its ith row.

b) Show that if the matrix $\mathbf{A} = [a_{ij}]$ is invertible (as defined in the preamble to Exercise 18), then the solution of the system in part (a) can be found using the equation $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

25. Use Exercises 18 and 24 to solve the system

$$\begin{aligned} 7x_1 - 8x_2 + 5x_3 &= 5 \\ -4x_1 + 5x_2 - 3x_3 &= -3 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}$$

26. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Find

a) $\mathbf{A} \vee \mathbf{B}$. b) $\mathbf{A} \wedge \mathbf{B}$. c) $\mathbf{A} \odot \mathbf{B}$.

27. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find

a) $\mathbf{A} \vee \mathbf{B}$. b) $\mathbf{A} \wedge \mathbf{B}$. c) $\mathbf{A} \odot \mathbf{B}$.

28. Find the Boolean product of $\mathbf{A}$ and $\mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

29. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find

a) $\mathbf{A}^{[2]}$. b) $\mathbf{A}^{[3]}$.
c) $\mathbf{A} \vee \mathbf{A}^{[2]} \vee \mathbf{A}^{[3]}$.

30. Let $\mathbf{A}$ be a zero–one matrix. Show that

a) $\mathbf{A} \vee \mathbf{A} = \mathbf{A}$. b) $\mathbf{A} \wedge \mathbf{A} = \mathbf{A}$.

31. In this exercise we show that the meet and join operations are commutative. Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ zero–one matrices. Show that

a) $\mathbf{A} \vee \mathbf{B} = \mathbf{B} \vee \mathbf{A}$. b) $\mathbf{B} \wedge \mathbf{A} = \mathbf{A} \wedge \mathbf{B}$.

32. In this exercise we show that the meet and join operations are associative. Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be $m \times n$ zero–one matrices. Show that

a) $(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C} = \mathbf{A} \vee (\mathbf{B} \vee \mathbf{C})$.
b) $(\mathbf{A} \wedge \mathbf{B}) \wedge \mathbf{C} = \mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C})$.

33. We will establish distributive laws of the meet over the join operation in this exercise. Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be $m \times n$ zero–one matrices. Show that

a) $\mathbf{A} \vee (\mathbf{B} \wedge \mathbf{C}) = (\mathbf{A} \vee \mathbf{B}) \wedge (\mathbf{A} \vee \mathbf{C})$.
b) $\mathbf{A} \wedge (\mathbf{B} \vee \mathbf{C}) = (\mathbf{A} \wedge \mathbf{B}) \vee (\mathbf{A} \wedge \mathbf{C})$.

34. Let $\mathbf{A}$ be an $n \times n$ zero–one matrix. Let $\mathbf{I}$ be the $n \times n$ identity matrix. Show that $\mathbf{A} \odot \mathbf{I} = \mathbf{I} \odot \mathbf{A} = \mathbf{A}$.

35. In this exercise we will show that the Boolean product of zero–one matrices is associative. Assume that $\mathbf{A}$ is an $m \times p$ zero–one matrix, $\mathbf{B}$ is a $p \times k$ zero–one matrix, and $\mathbf{C}$ is a $k \times n$ zero–one matrix. Show that $\mathbf{A} \odot (\mathbf{B} \odot \mathbf{C}) = (\mathbf{A} \odot \mathbf{B}) \odot \mathbf{C}$.

Key Terms and Results

TERMS

set: a collection of distinct objects

axiom: a basic assumption of a theory

paradox: a logical inconsistency

element, member of a set: an object in a set

roster method: a method that describes a set by listing its elements

set builder notation: the notation that describes a set by stating a property an element must have to be a member

∅ (empty set, null set): the set with no members

universal set: the set containing all objects under consideration

Venn diagram: a graphical representation of a set or sets

$S = T$ (set equality): S and T have the same elements

$S \subseteq T$ (S is a subset of T): every element of S is also an element of T

$S \subset T$ (S is a proper subset of T): S is a subset of T and $S \neq T$

finite set: a set with n elements, where n is a nonnegative integer

infinite set: a set that is not finite

$|S|$ (the cardinality of S): the number of elements in S

$P(S)$ (the power set of S): the set of all subsets of S

$A \cup B$ (the union of A and B): the set containing those elements that are in at least one of A and B

$A \cap B$ (the intersection of A and B): the set containing those elements that are in both A and B.