We can prove combinatorial identities by counting bit strings with different properties, as the proof of Theorem 4 will demonstrate.

THEOREM 4 Let n and r be nonnegative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Proof: We use a combinatorial proof. By Example 14 in Section 6.3, the left-hand side, $\binom{n+1}{r+1}$, counts the bit strings of length $n + 1$ containing $r + 1$ ones.

We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r + 1$ ones. This final one must occur at position $r + 1, r + 2, \ldots,$ or $n + 1$. Furthermore, if the last one is the kth bit there must be r ones among the first $k - 1$ positions. Consequently, by Example 14 in Section 6.3, there are $\binom{k-1}{r}$ such bit strings. Summing over k with $r + 1 \leq k \leq n + 1$, we find that there are

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

bit strings of length n containing exactly $r + 1$ ones. (Note that the last step follows from the change of variables $j = k - 1$.) Because the left-hand side and the right-hand side count the same objects, they are equal. This completes the proof. ◁

Exercises

1. Find the expansion of $(x + y)^4$
   a) using combinatorial reasoning, as in Example 1.
   b) using the binomial theorem.
2. Find the expansion of $(x + y)^5$
   a) using combinatorial reasoning, as in Example 1.
   b) using the binomial theorem.
3. Find the expansion of $(x + y)^6$.
4. Find the coefficient of x^5y^8 in $(x + y)^{13}$.
5. How many terms are there in the expansion of $(x + y)^{100}$ after like terms are collected?
6. What is the coefficient of x^7 in $(1 + x)^{11}$?
7. What is the coefficient of x^9 in $(2 - x)^{19}$?
8. What is the coefficient of x^8y^9 in the expansion of $(3x + 2y)^{17}$?
9. What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$?

*10. Give a formula for the coefficient of x^k in the expansion of $(x + 1/x)^{100}$, where k is an integer.

*11. Give a formula for the coefficient of x^k in the expansion of $(x^2 - 1/x)^{100}$, where k is an integer.

12. The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \leq k \leq 10$, is:

    1 10 45 120 210 252 210 120 45 10 1

    Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.
13. What is the row of Pascal's triangle containing the binomial coefficients $\binom{9}{k}$, $0 \leq k \leq 9$?
14. Show that if n is a positive integer, then $1 = \binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n} = 1$.
15. Show that $\binom{n}{k} \leq 2^n$ for all positive integers n and all integers k with $0 \leq k \leq n$.
16. a) Use Exercise 14 and Corollary 1 to show that if n is an integer greater than 1, then $\binom{n}{\lfloor n/2 \rfloor} \geq 2^n/n$.
    b) Conclude from part (a) that if n is a positive integer, then $\binom{2n}{n} \geq 4^n/2n$.
17. Show that if n and k are integers with $1 \leq k \leq n$, then $\binom{n}{k} \leq n^k/2^{k-1}$.
18. Suppose that b is an integer with $b \geq 7$. Use the binomial theorem and the appropriate row of Pascal's triangle to find the base-b expansion of $(11)_b^4$ [that is, the fourth power of the number $(11)_b$ in base-b notation].
19. Prove Pascal's identity, using the formula for $\binom{n}{r}$.
20. Suppose that k and n are integers with $1 \leq k < n$. Prove the **hexagon identity**

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1},$$

which relates terms in Pascal's triangle that form a hexagon.

21. Prove that if n and k are integers with $1 \le k \le n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$,

a) using a combinatorial proof. [*Hint:* Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.]

b) using an algebraic proof based on the formula for $\binom{n}{r}$ given in Theorem 2 in Section 6.3.

22. Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and k are nonnegative integers with $r \le n$ and $k \le r$,

a) using a combinatorial argument.

b) using an argument based on the formula for the number of r-combinations of a set with n elements.

23. Show that if n and k are positive integers, then

$$\binom{n+1}{k} = (n+1)\binom{n}{k-1} \Big/ k.$$

Use this identity to construct an inductive definition of the binomial coefficients.

24. Show that if p is a prime and k is an integer such that $1 \le k \le p-1$, then p divides $\binom{p}{k}$.

25. Let n be a positive integer. Show that

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+2}{n+1}\Big/2.$$

***26.** Let n and k be integers with $1 \le k \le n$. Show that

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \binom{2n+2}{n+1}\Big/2 - \binom{2n}{n}.$$

***27.** Prove the **hockeystick identity**

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

a) using a combinatorial argument.

b) using Pascal's identity.

28. Show that if n is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$

a) using a combinatorial argument.

b) by algebraic manipulation.

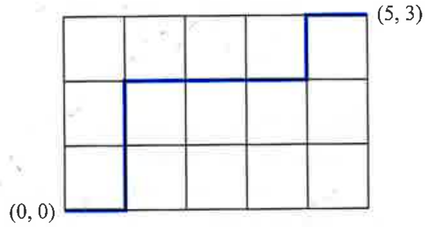
***29.** Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$. [*Hint:* Count in two ways the number of ways to select a committee and to then select a leader of the committee.]

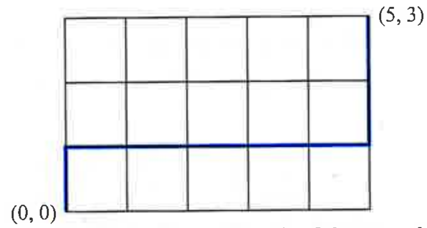
***30.** Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$. [*Hint:* Count in two ways the number of ways to select a committee, with n members from a group of n mathematics professors and n computer science professors, such that the chairperson of the committee is a mathematics professor.]

31. Show that a nonempty set has the same number of subsets with an odd number of elements as it does subsets with an even number of elements.

***32.** Prove the binomial theorem using mathematical induction.

33. In this exercise we will count the number of paths in the xy plane between the origin $(0, 0)$ and point (m, n), where m and n are nonnegative integers, such that each path is made up of a series of steps, where each step is a move one unit to the right or a move one unit upward. (No moves to the left or downward are allowed.) Two such paths from $(0, 0)$ to $(5, 3)$ are illustrated here.





a) Show that each path of the type described can be represented by a bit string consisting of m 0s and n 1s, where a 0 represents a move one unit to the right and a 1 represents a move one unit upward.

b) Conclude from part (a) that there are $\binom{m+n}{n}$ paths of the desired type.

34. Use Exercise 33 to give an alternative proof of Corollary 2 in Section 6.3, which states that $\binom{n}{k} = \binom{n}{n-k}$ whenever k is an integer with $0 \le k \le n$. [*Hint:* Consider the number of paths of the type described in Exercise 33 from $(0, 0)$ to $(n-k, k)$ and from $(0, 0)$ to $(k, n-k)$.]

35. Use Exercise 33 to prove Theorem 4. [*Hint:* Count the number of paths with n steps of the type described in Exercise 33. Every such path must end at one of the points $(n-k, k)$ for $k = 0, 1, 2, \ldots, n$.]

36. Use Exercise 33 to prove Pascal's identity. [*Hint:* Show that a path of the type described in Exercise 33 from $(0, 0)$ to $(n+1-k, k)$ passes through either $(n+1-k, k-1)$ or $(n-k, k)$, but not through both.]

37. Use Exercise 33 to prove the hockeystick identity from Exercise 27. [*Hint:* First, note that the number of paths from $(0, 0)$ to $(n+1, r)$ equals $\binom{n+1+r}{r}$. Second, count the number of paths by summing the number of these paths that start by going k units upward for $k = 0, 1, 2, \ldots, r$.]

38. Give a combinatorial proof that if n is a positive integer then $\sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}$. [*Hint:* Show that both sides count the ways to select a subset of a set of n elements together with two not necessarily distinct elements from this subset. Furthermore, express the right-hand side as $n(n-1)2^{n-2} + n2^{n-1}$.]

***39.** Determine a formula involving binomial coefficients for the nth term of a sequence if its initial terms are those listed. [*Hint:* Looking at Pascal's triangle will be helpful.

Although infinitely many sequences start with a specified set of terms, each of the following lists is the start of a sequence of the type desired.]

a) 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, ...
b) 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, ...
c) 1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, ...
d) 1, 1, 2, 3, 6, 10, 20, 35, 70, 126, ...
e) 1, 1, 1, 3, 1, 5, 15, 35, 1, 9, ...
f) 1, 3, 15, 84, 495, 3003, 18564, 116280, 735471, 4686825, ...

6.5 Generalized Permutations and Combinations

Introduction

In many counting problems, elements may be used repeatedly. For instance, a letter or digit may be used more than once on a license plate. When a dozen donuts are selected, each variety can be chosen repeatedly. This contrasts with the counting problems discussed earlier in the chapter where we considered only permutations and combinations in which each item could be used at most once. In this section we will show how to solve counting problems where elements may be used more than once.

Also, some counting problems involve indistinguishable elements. For instance, to count the number of ways the letters of the word *SUCCESS* can be rearranged, the placement of identical letters must be considered. This contrasts with the counting problems discussed earlier where all elements were considered distinguishable. In this section we will describe how to solve counting problems in which some elements are indistinguishable.

Moreover, in this section we will explain how to solve another important class of counting problems, problems involving counting the ways distinguishable elements can be placed in boxes. An example of this type of problem is the number of different ways poker hands can be dealt to four players.

Taken together, the methods described earlier in this chapter and the methods introduced in this section form a useful toolbox for solving a wide range of counting problems. When the additional methods discussed in Chapter 8 are added to this arsenal, you will be able to solve a large percentage of the counting problems that arise in a wide range of areas of study.

Permutations with Repetition

Counting permutations when repetition of elements is allowed can easily be done using the product rule, as Example 1 shows.

EXAMPLE 1 How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: By the product rule, because there are 26 uppercase English letters, and because each letter can be used repeatedly, we see that there are 26^r strings of uppercase English letters of length r. ◀

The number of r-permutations of a set with n elements when repetition is allowed is given in Theorem 1.

THEOREM 1 The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed, because for each choice all n objects are available. Hence, by the product rule there are n^r r-permutations when repetition is allowed. ◁