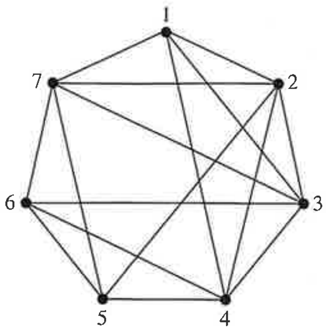
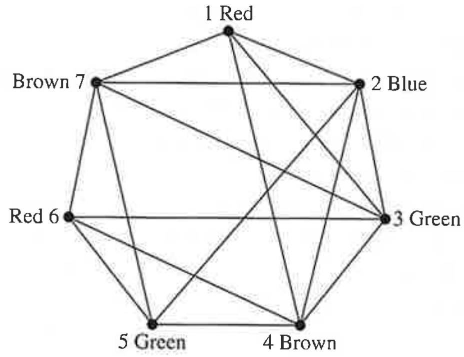


FIGURE 8 **The Graph Representing the Scheduling of Final Exams.**



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

FIGURE 9 **Using a Coloring to Schedule Final Exams.**

Now consider an application to the assignment of television channels.

EXAMPLE 6 **Frequency Assignments** Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled by graph coloring?

Solution: Construct a graph by assigning a vertex to each station. Two vertices are connected by an edge if they are located within 150 miles of each other. An assignment of channels corresponds to a coloring of the graph, where each color represents a different channel. ◂

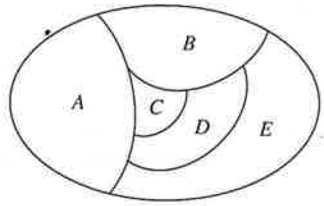
An application of graph coloring to compilers is considered in Example 7.

EXAMPLE 7 **Index Registers** In efficient compilers the execution of loops is speeded up when frequently used variables are stored temporarily in index registers in the central processing unit, instead of in regular memory. For a given loop, how many index registers are needed? This problem can be addressed using a graph coloring model. To set up the model, let each vertex of a graph represent a variable in the loop. There is an edge between two vertices if the variables they represent must be stored in index registers at the same time during the execution of the loop. Thus, the chromatic number of the graph gives the number of index registers needed, because different registers must be assigned to variables when the vertices representing these variables are adjacent in the graph. ◂

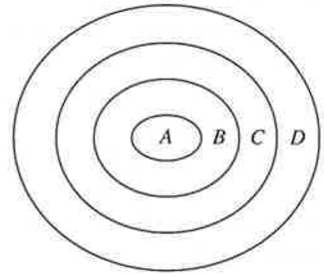
Exercises

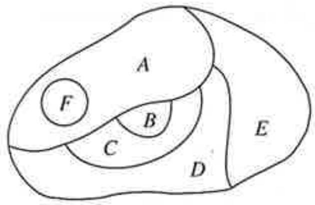
In Exercises 1–4 construct the dual graph for the map shown. Then find the number of colors needed to color the map so that no two adjacent regions have the same color.

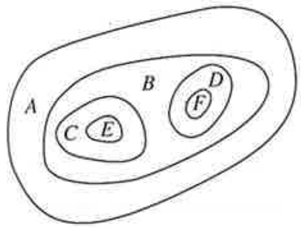
1.



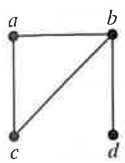
2.

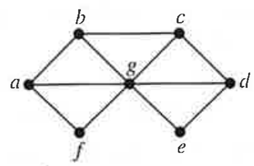


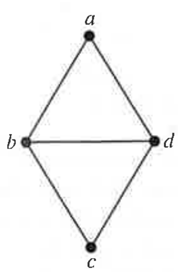
3. 

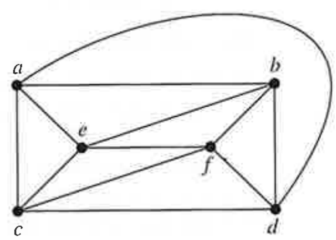
4. 

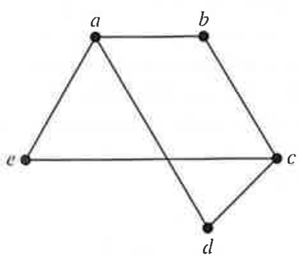
In Exercises 5–11 find the chromatic number of the given graph.

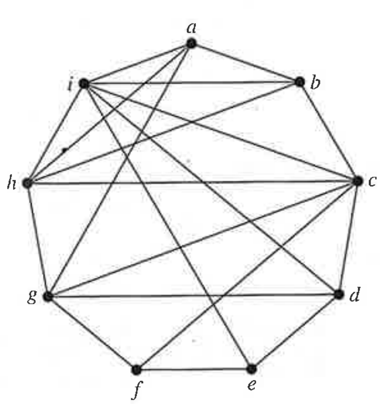
5. 

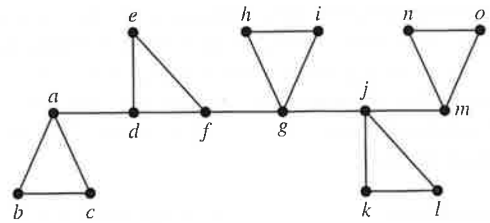
6. 

7. 

8. 

9. 

10. 

11. 

12. For the graphs in Exercises 5–11, decide whether it is possible to decrease the chromatic number by removing a single vertex and all edges incident with it.

13. Which graphs have a chromatic number of 1?

14. What is the least number of colors needed to color a map of the United States? Do not consider adjacent states that meet only at a corner. Suppose that Michigan is one region. Consider the vertices representing Alaska and Hawaii as isolated vertices.

15. What is the chromatic number of W_n?

16. Show that a simple graph that has a circuit with an odd number of vertices in it cannot be colored using two colors.

17. Schedule the final exams for Math 115, Math 116, Math 185, Math 195, CS 101, CS 102, CS 273, and CS 473, using the fewest number of different time slots, if there are no students taking both Math 115 and CS 473, both Math 116 and CS 473, both Math 195 and CS 101, both Math 195 and CS 102, both Math 115 and Math 116, both Math 115 and Math 185, and both Math 185 and Math 195, but there are students in every other pair of courses.

18. How many different channels are needed for six stations located at the distances shown in the table, if two stations cannot use the same channel when they are within 150 miles of each other?

	1	*2*	*3*	*4*	*5*	*6*
1	—	85	175	200	50	100
2	85	—	125	175	100	160
3	175	125	—	100	200	250
4	200	175	100	—	210	220
5	50	100	200	210	—	100
6	100	160	250	220	100	—

19. The mathematics department has six committees, each meeting once a month. How many different meeting times must be used to ensure that no member is scheduled to attend two meetings at the same time if the committees are C_1 = {Arlinghaus, Brand, Zaslavsky}, C_2 = {Brand, Lee, Rosen}, C_3 = {Arlinghaus, Rosen, Zaslavsky}, C_4 = {Lee, Rosen, Zaslavsky}, C_5 = {Arlinghaus, Brand}, and C_6 = {Brand, Rosen, Zaslavsky}?

20. A zoo wants to set up natural habitats in which to exhibit its animals. Unfortunately, some animals will eat some of the others when given the opportunity. How can a graph model and a coloring be used to determine the number of different habitats needed and the placement of the animals in these habitats?

An **edge coloring** of a graph is an assignment of colors to edges so that edges incident with a common vertex are assigned different colors. The **edge chromatic number** of a graph is the smallest number of colors that can be used in an edge coloring of the graph. The edge chromatic number of a graph G is denoted by $\chi'(G)$.

21. Find the edge chromatic number of each of the graphs in Exercises 5–11.

22. Suppose that n devices are on a circuit board and that these devices are connected by colored wires. Express the number of colors needed for the wires, in terms of the edge chromatic number of the graph representing this circuit board, under the requirement that the wires leaving a particular device must be different colors. Explain your answer.

23. Find the edge chromatic numbers of

a) C_n, where $n \geq 3$.

b) W_n, where $n \geq 3$.

24. Show that the edge chromatic number of a graph must be at least as large as the maximum degree of a vertex of the graph.

25. Show that if G is a graph with n vertices, then no more than $n/2$ edges can be colored the same in an edge coloring of G.

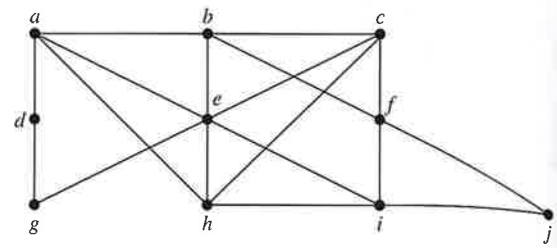
***26.** Find the edge chromatic number of K_n when n is a positive integer.

27. Seven variables occur in a loop of a computer program. The variables and the steps during which they must be stored are t: steps 1 through 6; u: step 2; v: steps 2 through 4; w: steps 1, 3, and 5; x: steps 1 and 6; y: steps 3 through 6; and z: steps 4 and 5. How many different index registers are needed to store these variables during execution?

28. What can be said about the chromatic number of a graph that has K_n as a subgraph?

This algorithm can be used to color a simple graph: First, list the vertices $v_1, v_2, v_3, \ldots, v_n$ in order of decreasing degree so that $\deg(v_1) \geq \deg(v_2) \geq \cdots \geq \deg(v_n)$. Assign color 1 to v_1 and to the next vertex in the list not adjacent to v_1 (if one exists), and successively to each vertex in the list not adjacent to a vertex already assigned color 1. Then assign color 2 to the first vertex in the list not already colored. Successively assign color 2 to vertices in the list that have not already been colored and are not adjacent to vertices assigned color 2. If uncolored vertices remain, assign color 3 to the first vertex in the list not yet colored, and use color 3 to successively color those vertices not already colored and not adjacent to vertices assigned color 3. Continue this process until all vertices are colored.

29. Construct a coloring of the graph shown using this algorithm.



***30.** Use pseudocode to describe this coloring algorithm.

***31.** Show that the coloring produced by this algorithm may use more colors than are necessary to color a graph.

A connected graph G is called **chromatically k-critical** if the chromatic number of G is k, but for every edge of G, the chromatic number of the graph obtained by deleting this edge from G is $k-1$.

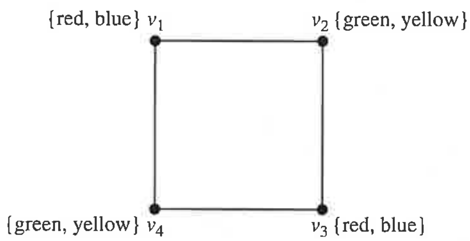
32. Show that C_n is chromatically 3-critical whenever n is an odd positive integer, $n \geq 3$.

33. Show that W_n is chromatically 4-critical whenever n is an odd integer, $n \geq 3$.

34. Show that W_4 is not chromatically 3-critical.

35. Show that if G is a chromatically k-critical graph, then the degree of every vertex of G is at least $k-1$.

A **k-tuple coloring** of a graph G is an assignment of a set of k different colors to each of the vertices of G such that no two adjacent vertices are assigned a common color. We denote by $\chi_k(G)$ the smallest positive integer n such that G has a k-tuple coloring using n colors. For example, $\chi_2(C_4) = 4$. To see this, note that using only four colors we can assign two colors to each vertex of C_4, as illustrated, so that no two adjacent vertices are assigned the same color. Furthermore, no fewer than four colors suffice because the vertices v_1 and v_2 each must be assigned two colors, and a common color cannot be assigned to both v_1 and v_2. (For more information about k-tuple coloring, see [MiRo91].)



36. Find these values:

a) $\chi_2(K_3)$ **b)** $\chi_2(K_4)$ **c)** $\chi_2(W_4)$

d) $\chi_2(C_5)$ **e)** $\chi_2(K_{3,4})$ **f)** $\chi_3(K_5)$

***g)** $\chi_3(C_5)$ **h)** $\chi_3(K_{4,5})$

*37. Let G and H be the graphs displayed in Figure 3. Find

a) $\chi_2(G)$. b) $\chi_2(H)$.
c) $\chi_3(G)$. d) $\chi_3(H)$.

38. What is $\chi_k(G)$ if G is a bipartite graph and k is a positive integer?

39. Frequencies for mobile radio (or cellular) telephones are assigned by zones. Each zone is assigned a set of frequencies to be used by vehicles in that zone. The same frequency cannot be used in different zones when interference can occur between telephones in these zones. Explain how a k-tuple coloring can be used to assign k frequencies to each mobile radio zone in a region.

*40. Show that every planar graph G can be colored using six or fewer colors. [*Hint:* Use mathematical induction on the number of vertices of the graph. Apply Corollary 2 of Section 10.7 to find a vertex v with $\deg(v) \leq 5$. Consider the subgraph of G obtained by deleting v and all edges incident with it.]

**41. Show that every planar graph G can be colored using five or fewer colors. [*Hint:* Use the hint provided for Exercise 40.]

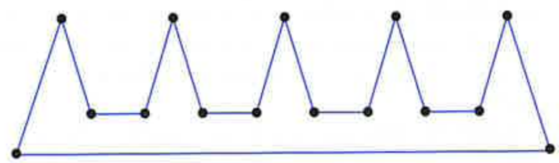
The famous Art Gallery Problem asks how many guards are needed to see all parts of an art gallery, where the gallery is the interior and boundary of a polygon with n sides. To state this problem more precisely, we need some terminology. A point x inside or on the boundary of a simple polygon P **covers** or **sees** a point y inside or on P if all points on the line segment xy are in the interior or on the boundary of P. We say that a set of points is a **guarding set** of a simple polygon P if for every point y inside P or on the boundary of P there is a point x in this guarding set that sees y. Denote by $G(P)$ the minimum number of points needed to guard the simple polygon P. The **art gallery problem** asks for the function $g(n)$, which is the maximum value of $G(P)$ over all simple polygons with n vertices. That is, $g(n)$ is the minimum positive integer for which it is guaranteed that a simple polygon with n vertices can be guarded with $g(n)$ or fewer guards.

42. Show that $g(3) = 1$ and $g(4) = 1$ by showing that all triangles and quadrilaterals can be guarded using one point.

*43. Show that $g(5) = 1$. That is, show that all pentagons can be guarded using one point. [*Hint:* Show that there are either 0, 1, or 2 vertices with an interior angle greater than 180 degrees and that in each case, one guard suffices.]

*44. Show that $g(6) = 2$ by first using Exercises 42 and 43 as well as Lemma 1 in Section 5.2 to show that $g(6) \leq 2$ and then find a simple hexagon for which two guards are needed.

*45. Show that $g(n) \geq \lfloor n/3 \rfloor$. [*Hint:* Consider the polygon with $3k$ vertices that resembles a comb with k prongs, such as the polygon with 15 sides shown here.]



*46. Solve the art gallery problem by proving the **art gallery theorem**, which states that at most $\lfloor n/3 \rfloor$ guards are needed to guard the interior and boundary of a simple polygon with n vertices. [*Hint:* Use Theorem 1 in Section 5.2 to triangulate the simple polygon into $n - 2$ triangles. Then show that it is possible to color the vertices of the triangulated polygon using three colors so that no two adjacent vertices have the same color. Use induction and Exercise 23 in Section 5.2. Finally, put guards at all vertices that are colored red, where red is the color used least in the coloring of the vertices. Show that placing guards at these points is all that is needed.]

Key Terms and Results

TERMS

undirected edge: an edge associated to a set $\{u, v\}$, where u and v are vertices

directed edge: an edge associated to an ordered pair (u, v), where u and v are vertices

multiple edges: distinct edges connecting the same vertices

multiple directed edges: distinct directed edges associated with the same ordered pair (u, v), where u and v are vertices

loop: an edge connecting a vertex with itself

undirected graph: a set of vertices and a set of undirected edges each of which is associated with a set of one or two of these vertices

simple graph: an undirected graph with no multiple edges or loops

multigraph: an undirected graph that may contain multiple edges but no loops

pseudograph: an undirected graph that may contain multiple edges and loops

directed graph: a set of vertices together with a set of directed edges each of which is associated with an ordered pair of vertices

directed multigraph: a graph with directed edges that may contain multiple directed edges

simple directed graph: a directed graph without loops or multiple directed edges

adjacent: two vertices are adjacent if there is an edge between them

incident: an edge is incident with a vertex if the vertex is an endpoint of that edge

deg v (degree of the vertex v in an undirected graph): the number of edges incident with v with loops counted twice