EXAMPLE 12 Determine whether 2 and 3 are primitive roots modulo 11.

Solution: When we compute the powers of 2 in $\mathbf{Z}_{11}$, we obtain $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 5, 2^5 = 10, 2^6 = 9, 2^7 = 7, 2^8 = 3, 2^9 = 6, 2^{10} = 1$. Because every element of $\mathbf{Z}_{11}$ is a power of 2, 2 is a primitive root of 11.

When we compute the powers of 3 modulo 11, we obtain $3^1 = 3, 3^2 = 9, 3^3 = 5, 3^4 = 4, 3^5 = 1$. We note that this pattern repeats when we compute higher powers of 3. Because not all elements of $\mathbf{Z}_{11}$ are powers of 3, we conclude that 3 is not a primitive root of 11. ◀

An important fact in number theory is that there is a primitive root modulo p for every prime p. We refer the reader to [Ro10] for a proof of this fact. Suppose that p is prime and r is a primitive root modulo p. If a is an integer between 1 and $p - 1$, that is, an element of $\mathbf{Z}_p$, we know that there is an unique exponent e such that $r^e = a$ in $\mathbf{Z}_p$, that is, $r^e \bmod p = a$.

DEFINITION 4 Suppose that p is a prime, r is a primitive root modulo p, and a is an integer between 1 and $p - 1$ inclusive. If $r^e \bmod p = a$ and $0 \leq e \leq p - 1$, we say that e is the *discrete logarithm* of a modulo p to the base r and we write $\log_r a = e$ (where the prime p is understood).

EXAMPLE 13 Find the discrete logarithms of 3 and 5 modulo 11 to the base 2.

Solution: When we computed the powers of 2 modulo 11 in Example 12, we found that $2^8 = 3$ and $2^4 = 5$ in $\mathbf{Z}_{11}$. Hence, the discrete logarithms of 3 and 5 modulo 11 to the base 2 are 8 and 4, respectively. (These are the powers of 2 that equal 3 and 5, respectively, in $\mathbf{Z}_{11}$.) We write $\log_2 3 = 8$ and $\log_2 5 = 4$ (where the modulus 11 is understood and not explicitly noted in the notation). ◀

The discrete logarithm problem is hard!

The **discrete logarithm problem** takes as input a prime p, a primitive root r modulo p, and a positive integer $a \in \mathbf{Z}_p$; its output is the discrete logarithm of a modulo p to the base r. Although this problem might seem not to be that difficult, it turns out that no polynomial time algorithm is known for solving it. The difficulty of this problem plays an important role in cryptography, as we will see in Section 4.6

Exercises

1. Show that 15 is an inverse of 7 modulo 26.
2. Show that 937 is an inverse of 13 modulo 2436.
3. By inspection (as discussed prior to Example 1), find an inverse of 4 modulo 9.
4. By inspection (as discussed prior to Example 1), find an inverse of 2 modulo 17.
5. Find an inverse of a modulo m for each of these pairs of relatively prime integers using the method followed in Example 2.
   a) $a = 4, m = 9$
   b) $a = 19, m = 141$
   c) $a = 55, m = 89$
   d) $a = 89, m = 232$
6. Find an inverse of a modulo m for each of these pairs of relatively prime integers using the method followed in Example 2.
   a) $a = 2, m = 17$
   b) $a = 34, m = 89$
   c) $a = 144, m = 233$
   d) $a = 200, m = 1001$
7. Show that if a and m are relatively prime positive integers, then the inverse of a modulo m is unique modulo m. [*Hint:* Assume that there are two solutions b and c of the congruence $ax \equiv 1 \pmod m$. Use Theorem 7 of Section 4.3 to show that $b \equiv c \pmod m$.]
8. Show that an inverse of a modulo m, where a is an integer and $m > 2$ is a positive integer, does not exist if $\gcd(a, m) > 1$.
9. Solve the congruence $4x \equiv 5 \pmod 9$ using the inverse of 4 modulo 9 found in part (a) of Exercise 5.
10. Solve the congruence $2x \equiv 7 \pmod{17}$ using the inverse of 2 modulo 7 found in part (a) of Exercise 6.
11. Solve each of these congruences using the modular inverses found in parts (b), (c), and (d) of Exercise 5.
    a) $19x \equiv 4 \pmod{141}$
    b) $55x \equiv 34 \pmod{89}$
    c) $89x \equiv 2 \pmod{232}$

12. Solve each of these congruences using the modular inverses found in parts (b), (c), and (d) of Exercise 6.
   a) $34x \equiv 77 \pmod{89}$
   b) $144x \equiv 4 \pmod{233}$
   c) $200x \equiv 13 \pmod{1001}$

13. Find the solutions of the congruence $15x^2 + 19x \equiv 5 \pmod{11}$. [*Hint:* Show the congruence is equivalent to the congruence $15x^2 + 19x + 6 \equiv 0 \pmod{11}$. Factor the left-hand side of the congruence; show that a solution of the quadratic congruence is a solution of one of the two different linear congruences.]

14. Find the solutions of the congruence $12x^2 + 25x \equiv 10 \pmod{11}$. [*Hint:* Show the congruence is equivalence to the congruence $12x^2 + 25x + 12 \equiv 0 \pmod{11}$. Factor the left-hand side of the congruence; show that a solution of the quadratic congruence is a solution of one of two different linear congruences.]

***15.** Show that if m is an integer greater than 1 and $ac \equiv bc \pmod{m}$, then $a \equiv b \pmod{m/\gcd(c, m)}$.

16. **a)** Show that the positive integers less than 11, except 1 and 10, can be split into pairs of integers such that each pair consists of integers that are inverses of each other modulo 11.
   b) Use part (a) to show that $10! \equiv -1 \pmod{11}$.

17. Show that if p is prime, the only solutions of $x^2 \equiv 1 \pmod{p}$ are integers x such that $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$.

***18.** **a)** Generalize the result in part (a) of Exercise 16; that is, show that if p is a prime, the positive integers less than p, except 1 and $p - 1$, can be split into $(p - 3)/2$ pairs of integers such that each pair consists of integers that are inverses of each other. [*Hint:* Use the result of Exercise 17.]
   b) From part (a) conclude that $(p - 1)! \equiv -1 \pmod{p}$ whenever p is prime. This result is known as **Wilson's theorem**.
   c) What can we conclude if n is a positive integer such that $(n - 1)! \not\equiv -1 \pmod{n}$?

***19.** This exercise outlines a proof of Fermat's little theorem.
   a) Suppose that a is not divisible by the prime p. Show that no two of the integers $1 \cdot a, 2 \cdot a, \ldots, (p - 1)a$ are congruent modulo p.
   b) Conclude from part (a) that the product of $1, 2, \ldots, p - 1$ is congruent modulo p to the product of $a, 2a, \ldots, (p - 1)a$. Use this to show that

$$(p - 1)! \equiv a^{p-1}(p - 1)! \pmod{p}.$$

   c) Use Theorem 7 of Section 4.3 to show from part (b) that $a^{p-1} \equiv 1 \pmod{p}$ if $p \not| a$. [*Hint:* Use Lemma 3 of Section 4.3 to show that p does not divide $(p - 1)!$ and then use Theorem 7 of Section 4.3. Alternatively, use Wilson's theorem from Exercise 18(b).]
   d) Use part (c) to show that $a^p \equiv a \pmod{p}$ for all integers a.

20. Use the construction in the proof of the Chinese remainder theorem to find all solutions to the system of congruences $x \equiv 2 \pmod{3}$, $x \equiv 1 \pmod{4}$, and $x \equiv 3 \pmod{5}$.

21. Use the construction in the proof of the Chinese remainder theorem to find all solutions to the system of congruences $x \equiv 1 \pmod{2}$, $x \equiv 2 \pmod{3}$, $x \equiv 3 \pmod{5}$, and $x \equiv 4 \pmod{11}$.

22. Solve the system of congruence $x \equiv 3 \pmod{6}$ and $x \equiv 4 \pmod{7}$ using the method of back substitution.

23. Solve the system of congruences in Exercise 20 using the method of back substitution.

24. Solve the system of congruences in Exercise 21 using the method of back substitution.

25. Write out in pseudocode an algorithm for solving a simultaneous system of linear congruences based on the construction in the proof of the Chinese remainder theorem.

***26.** Find all solutions, if any, to the system of congruences $x \equiv 5 \pmod{6}$, $x \equiv 3 \pmod{10}$, and $x \equiv 8 \pmod{15}$.

***27.** Find all solutions, if any, to the system of congruences $x \equiv 7 \pmod{9}$, $x \equiv 4 \pmod{12}$, and $x \equiv 16 \pmod{21}$.

28. Use the Chinese remainder theorem to show that an integer a, with $0 \leq a < m = m_1 m_2 \cdots m_n$, where the positive integers $m_1, m_2, \ldots, m_n$ are pairwise relatively prime, can be represented uniquely by the n-tuple $(a \bmod m_1, a \bmod m_2, \ldots, a \bmod m_n)$.

***29.** Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime integers greater than or equal to 2. Show that if $a \equiv b \pmod{m_i}$ for $i = 1, 2, \ldots, n$, then $a \equiv b \pmod{m}$, where $m = m_1 m_2 \cdots m_n$. (This result will be used in Exercise 30 to prove the Chinese remainder theorem. Consequently, do not use the Chinese remainder theorem to prove it.)

***30.** Complete the proof of the Chinese remainder theorem by showing that the simultaneous solution of a system of linear congruences modulo pairwise relatively prime moduli is unique modulo the product of these moduli. [*Hint:* Assume that x and y are two simultaneous solutions. Show that $m_i \mid x - y$ for all i. Using Exercise 29, conclude that $m = m_1 m_2 \cdots m_n \mid x - y$.]

31. Which integers leave a remainder of 1 when divided by 2 and also leave a remainder of 1 when divided by 3?

32. Which integers are divisible by 5 but leave a remainder of 1 when divided by 3?

33. Use Fermat's little theorem to find $7^{121} \bmod 13$.

34. Use Fermat's little theorem to find $23^{1002} \bmod 41$.

35. Use Fermat's little theorem to show that if p is prime and $p \not| a$, then a^{p-2} is an inverse of a modulo p.

36. Use Exercise 35 to find an inverse of 5 modulo 41.

37. **a)** Show that $2^{340} \equiv 1 \pmod{11}$ by Fermat's little theorem and noting that $2^{340} = (2^{10})^{34}$.
   b) Show that $2^{340} \equiv 1 \pmod{31}$ using the fact that $2^{340} = (2^5)^{68} = 32^{68}$.
   c) Conclude from parts (a) and (b) that $2^{340} \equiv 1 \pmod{341}$.

38. **a)** Use Fermat's little theorem to compute 3^{302} **mod** 5, 3^{302} **mod** 7, and 3^{302} **mod** 11.

b) Use your results from part (a) and the Chinese remainder theorem to find 3^{302} **mod** 385. (Note that $385 = 5 \cdot 7 \cdot 11$.)

39. **a)** Use Fermat's little theorem to compute 5^{2003} **mod** 7, 5^{2003} **mod** 11, and 5^{2003} **mod** 13.

b) Use your results from part (a) and the Chinese remainder theorem to find 5^{2003} **mod** 1001. (Note that $1001 = 7 \cdot 11 \cdot 13$.)

40. Show with the help of Fermat's little theorem that if n is a positive integer, then 42 divides $n^7 - n$.

41. Show that if p is an odd prime, then every divisor of the Mersenne number $2^p - 1$ is of the form $2kp + 1$, where k is a nonnegative integer. [*Hint:* Use Fermat's little theorem and Exercise 37 of Section 4.3.]

42. Use Exercise 41 to determine whether $M_{13} = 2^{13} - 1 = 8191$ and $M_{23} = 2^{23} - 1 = 8{,}388{,}607$ are prime.

43. Use Exercise 41 to determine whether $M_{11} = 2^{11} - 1 = 2047$ and $M_{17} = 2^{17} - 1 = 131{,}071$ are prime.

☞ Let n be a positive integer and let $n - 1 = 2^s t$, where s is a nonnegative integer and t is an odd positive integer. We say that n passes **Miller's test for the base b** if either $b^t \equiv 1 \pmod{n}$ or $b^{2^j t} \equiv -1 \pmod{n}$ for some j with $0 \leq j \leq s - 1$. It can be shown (see [Ro10]) that a composite integer n passes Miller's test for fewer than $n/4$ bases b with $1 < b < n$. A composite positive integer n that passes Miller's test to the base b is called a **strong pseudoprime to the base b**.

***44.** Show that if n is prime and b is a positive integer with $n \nmid b$, then n passes Miller's test to the base b.

45. Show that 2047 is a strong pseudoprime to the base 2 by showing that it passes Miller's test to the base 2, but is composite.

46. Show that 1729 is a Carmichael number.

47. Show that 2821 is a Carmichael number.

***48.** Show that if $n = p_1 p_2 \cdots p_k$, where $p_1, p_2, \ldots, p_k$ are distinct primes that satisfy $p_j - 1 \mid n - 1$ for $j = 1, 2, \ldots, k$, then n is a Carmichael number.

49. **a)** Use Exercise 48 to show that every integer of the form $(6m + 1)(12m + 1)(18m + 1)$, where m is a positive integer and $6m + 1$, $12m + 1$, and $18m + 1$ are all primes, is a Carmichael number.

b) Use part (a) to show that 172,947,529 is a Carmichael number.

50. Find the nonnegative integer a less than 28 represented by each of these pairs, where each pair represents (a **mod** 4, a **mod** 7).

a) (0, 0)	**b)** (1, 0)	**c)** (1, 1)
d) (2, 1)	**e)** (2, 2)	**f)** (0, 3)
g) (2, 0)	**h)** (3, 5)	**i)** (3, 6)

51. Express each nonnegative integer a less than 15 as a pair (a **mod** 3, a **mod** 5).

52. Explain how to use the pairs found in Exercise 51 to add 4 and 7.

53. Solve the system of congruences that arises in Example 8.

54. Show that 2 is a primitive root of 19.

55. Find the discrete logarithms of 5 and 6 to the base 2 modulo 19.

56. Let p be an odd prime and r a primitive root of p. Show that if a and b are positive integers in $\mathbf{Z}_p$, then $\log_r(ab) \equiv \log_r a + \log_r b \pmod{p-1}$.

57. Write out a table of discrete logarithms modulo 17 with respect to the primitive root 3.

If m is a positive integer, the integer a is a **quadratic residue** of m if $\gcd(a, m) = 1$ and the congruence $x^2 \equiv a \pmod{m}$ has a solution. In other words, a quadratic residue of m is an integer relatively prime to m that is a perfect square modulo m. If a is not a quadratic residue of m and $\gcd(a, m) = 1$, we say that it is a **quadratic nonresidue** of m. For example, 2 is a quadratic residue of 7 because $\gcd(2, 7) = 1$ and $3^2 \equiv 2 \pmod{7}$ and 3 is a quadratic nonresidue of 7 because $\gcd(3, 7) = 1$ and $x^2 \equiv 3 \pmod{7}$ has no solution.

58. Which integers are quadratic residues of 11?

59. Show that if p is an odd prime and a is an integer not divisible by p, then the congruence $x^2 \equiv a \pmod{p}$ has either no solutions or exactly two incongruent solutions modulo p.

60. Show that if p is an odd prime, then there are exactly $(p-1)/2$ quadratic residues of p among the integers $1, 2, \ldots, p - 1$.

If p is an odd prime and a is an integer not divisible by p, the **Legendre symbol** $\left(\frac{a}{p}\right)$ is defined to be 1 if a is a quadratic residue of p and -1 otherwise.

61. Show that if p is an odd prime and a and b are integers with $a \equiv b \pmod{p}$, then

$$\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right).$$

62. Prove **Euler's criterion**, which states that if p is an odd prime and a is a positive integer not divisible by p, then

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}.$$

[*Hint:* If a is a quadratic residue modulo p, apply Fermat's little theorem; otherwise, apply Wilson's theorem, given in Exercise 18(b).]

63. Use Exercise 62 to show that if p is an odd prime and a and b are integers not divisible by p, then

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right).$$

64. Show that if p is an odd prime, then -1 is a quadratic residue of p if $p \equiv 1 \pmod{4}$, and -1 is not a quadratic residue of p if $p \equiv 3 \pmod{4}$. [*Hint:* Use Exercise 62.]

65. Find all solutions of the congruence $x^2 \equiv 29 \pmod{35}$. [*Hint:* Find the solutions of this congruence modulo 5 and modulo 7, and then use the Chinese remainder theorem.]