These last two congruences hold because $\sum_{i=1}^{10} x_i \equiv 0 \pmod{10}$ and $11 \not| ja$, because $11 \not| j$ and $11 \not| a$. We conclude that $y_1 y_2 \ldots y_{10}$ is not a valid ISBN. So, we have detected the single error.

Now suppose that two unequal digits have been transposed. It follows that there are distinct integers j and k such that $y_j = x_k$ and $y_k = x_j$, and $y_i = x_i$ for $i \neq j$ and $i \neq k$. Hence,

$$\sum_{i=1}^{10} iy_i = \Big(\sum_{i=1}^{10} ix_i\Big) + (jx_k - jx_j) + (kx_j - kx_k) \equiv (j-k)(x_k - x_j) \not\equiv 0 \pmod{11},$$

because $\sum_{i=1}^{10} x_i \equiv 0 \pmod{10}$ and $11 \not| (j-k)$ and $11 \not| (x_k - x_j)$. We see that $y_1 y_2 \ldots y_{10}$ is not a valid ISBN. Thus, we can detect the interchange of two unequal digits.

Exercises

1. Which memory locations are assigned by the hashing function $h(k) = k$ **mod** 97 to the records of insurance company customers with these Social Security numbers?

a) 034567981 **b)** 183211232
c) 220195744 **d)** 987255335

2. Which memory locations are assigned by the hashing function $h(k) = k$ **mod** 101 to the records of insurance company customers with these Social Security numbers?

a) 104578690 **b)** 432222187
c) 372201919 **d)** 501338753

3. A parking lot has 31 visitor spaces, numbered from 0 to 30. Visitors are assigned parking spaces using the hashing function $h(k) = k$ **mod** 31, where k is the number formed from the first three digits on a visitor's license plate.

a) Which spaces are assigned by the hashing function to cars that have these first three digits on their license plates: 317, 918, 007, 100, 111, 310?

b) Describe a procedure visitors should follow to find a free parking space, when the space they are assigned is occupied.

Another way to resolve collisions in hashing is to use *double hashing*. We use an initial hashing function $h(k) = k$ **mod** p where p is prime. We also use a second hashing function $g(k) = (k+1)$ **mod** $(p-2)$. When a collision occurs, we use a *probing sequence* $h(k, i) = (h(k) + i \cdot g(k))$ **mod** p.

4. Use the double hashing procedure we have described with $p = 4969$ to assign memory locations to files for employees with social security numbers $k_1 = 132489971$, $k_2 = 509496993$, $k_3 = 546332190$, $k_4 = 034367980$, $k_5 = 047900151$, $k_6 = 329938157$, $k_7 = 212228844$, $k_8 = 325510778$, $k_9 = 353354519$, $k_{10} = 053708912$.

5. What sequence of pseudorandom numbers is generated using the linear congruential generator $x_{n+1} = (3x_n + 2)$ **mod** 13 with seed $x_0 = 1$?

6. What sequence of pseudorandom numbers is generated using the linear congruential generator $x_{n+1} = (4x_n + 1)$ **mod** 7 with seed $x_0 = 3$?

7. What sequence of pseudorandom numbers is generated using the pure multiplicative generator $x_{n+1} = 3x_n$ **mod** 11 with seed $x_0 = 2$?

8. Write an algorithm in pseudocode for generating a sequence of pseudorandom numbers using a linear congruential generator.

The **middle-square method** for generating pseudorandom numbers begins with an n-digit integer. This number is squared, initial zeros are appended to ensure that the result has $2n$ digits, and its middle n digits are used to form the next number in the sequence. This process is repeated to generate additional terms.

9. Find the first eight terms of the sequence of four-digit pseudorandom numbers generated by the middle square method starting with 2357.

10. Explain why both 3792 and 2916 would be bad choices for the initial term of a sequence of four-digit pseudorandom numbers generated by the middle square method.

The **power generator** is a method for generating pseudorandom numbers. To use the power generator, parameters p and d are specified, where p is a prime, d is a positive integer such that $p \not| d$, and a seed x_0 is specified. The pseudorandom numbers $x_1, x_2, \ldots$ are generated using the recursive definition $x_{n+1} = x_n^d$ **mod** p.

11. Find the sequence of pseudorandom numbers generated by the power generator with $p = 7$, $d = 3$, and seed $x_0 = 2$.

12. Find the sequence of pseudorandom numbers generated by the power generator with $p = 11$, $d = 2$, and seed $x_0 = 3$.

13. Suppose you received these bit strings over a communications link, where the last bit is a parity check bit. In which string are you sure there is an error?

a) 00000111111
b) 10101010101
c) 11111100000
d) 10111101111

14. Prove that a parity check bit can detect an error in a string if and only if the string contains an odd number of errors.

15. The first nine digits of the ISBN-10 of the European version of the fifth edition of this book are 0-07-119881. What is the check digit for that book?

16. The ISBN-10 of the sixth edition of *Elementary Number Theory and Its Applications* is 0-321-500Q1-8, where Q is a digit. Find the value of Q.

17. Determine whether the check digit of the ISBN-10 for this textbook (the seventh edition of *Discrete Mathematics and its Applications*) was computed correctly by the publisher.

The United States Postal Service (USPS) sells money orders identified by an 11-digit number $x_1x_2 \ldots x_{11}$. The first ten digits identify the money order; x_{11} is a check digit that satisfies $x_{11} = x_1 + x_2 + \cdots + x_{10}$ **mod** 9.

18. Find the check digit for the USPS money orders that have identification number that start with these ten digits.

a) 7555618873
b) 6966133421
c) 8018927435
d) 3289744134

19. Determine whether each of these numbers is a valid USPS money order identification number.

a) 74051489623
b) 88382013445
c) 56152240784
d) 66606631178

20. One digit in each of these identification numbers of a postal money order is smudged. Can you recover the smudged digit, indicated by a Q, in each of these numbers?

a) Q1223139784
b) 6702120Q988
c) 27Q41007734
d) 213279032Q1

21. One digit in each of these identification numbers of a postal money order is smudged. Can you recover the smudged digit, indicated by a Q, in each of these numbers?

a) 493212Q0688
b) 850Q9103858
c) 2Q941007734
d) 66687Q03201

22. Determine which single digit errors are detected by the USPS money order code.

23. Determine which transposition errors are detected by the USPS money order code.

24. Determine the check digit for the UPCs that have these initial 11 digits.

a) 73232184434
b) 63623991346
c) 04587320720
d) 93764323341

25. Determine whether each of the strings of 12 digits is a valid UPC code.

a) 036000291452
b) 012345678903
c) 782421843014
d) 726412175425

26. Does the check digit of a UPC code detect all single errors? Prove your answer or find a counterexample.

27. Determine which transposition errors the check digit of a UPC code finds.

Some airline tickets have a 15-digit identification number $a_1a_2 \ldots a_{15}$ where a_{15} is a check digit that equals $a_1a_2 \ldots a_{14}$ **mod** 7.

28. Find the check digit a_{15} that follows each of these initial 14 digits of an airline ticket identification number.

a) 10237424413392
b) 00032781811234
c) 00611232134231
d) 00193222543435

29. Determine whether each of these 15-digit numbers is a valid airline ticket identification number.

a) 101333341789013
b) 007862342770445
c) 113273438882531
d) 000122347322871

30. Which errors in a single digit of a 15-digit airline ticket identification number can be detected?

***31.** Can the accidental transposition of two consecutive digits in an airline ticket identification number be detected using the check digit?

Periodicals are identified using an **International Standard Serial Number (ISSN)**. An ISSN consists of two blocks of four digits. The last digit in the second block is a check digit. This check digit is determined by the congruence $d_8 \equiv 3d_1 + 4d_2 + 5d_3 + 6d_4 + 7d_5 + 8d_6 + 9d_7 \pmod{11}$. When $d_8 \equiv 10 \pmod{11}$, we use the letter X to represent d_8 in the code.

32. For each of these initial seven digits of an ISSN, determine the check digit (which may be the letter X).

a) 1570-868
b) 1553-734
c) 1089-708
d) 1383-811

33. Are each of these eight-digit codes possible ISSNs? That is, do they end with a correct check digit?

a) 1059-1027
b) 0002-9890
c) 1530-8669
d) 1007-120X

34. Does the check digit of an ISSN detect every single error in an ISSN? Justify your answer with either a proof or a counterexample.

35. Does the check digit of an ISSN detect every error where two consecutive digits are accidentally interchanged? Justify your answer with either a proof or a counterexample.