Exercises

1. Encrypt the message DO NOT PASS GO by translating the letters into numbers, applying the given encryption function, and then translating the numbers back into letters.
   a) $f(p) = (p + 3)$ **mod** 26 (the Caesar cipher)
   b) $f(p) = (p + 13)$ **mod** 26
   c) $f(p) = (3p + 7)$ **mod** 26

2. Encrypt the message STOP POLLUTION by translating the letters into numbers, applying the given encryption function, and then translating the numbers back into letters.
   a) $f(p) = (p + 4)$ **mod** 26
   b) $f(p) = (p + 21)$ **mod** 26
   c) $f(p) = (17p + 22)$ **mod** 26

3. Encrypt the message WATCH YOUR STEP by translating the letters into numbers, applying the given encryption function, and then translating the numbers back into letters.
   a) $f(p) = (p + 14)$ **mod** 26
   b) $f(p) = (14p + 21)$ **mod** 26
   c) $f(p) = (-7p + 1)$ **mod** 26

4. Decrypt these messages that were encrypted using the Caesar cipher.
   a) EOXH MHDQV
   b) WHVW WRGDB
   c) HDW GLP VXP

5. Decrypt these messages encrypted using the shift cipher $f(p) = (p + 10)$ **mod** 26.
   a) CEBBOXNOB XYG
   b) LO WI PBSOXN
   c) DSWO PYB PEX

6. Suppose that when a long string of text is encrypted using a shift cipher $f(p) = (p + k)$ **mod** 26, the most common letter in the ciphertext is X. What is the most likely value for k assuming that the distribution of letters in the text is typical of English text?

7. Suppose that when a string of English text is encrypted using a shift cipher $f(p) = (p + k)$ **mod** 26, the resulting ciphertext is DY CVOOZ ZOBMRKXMO DY NBOKW. What was the original plaintext string?

8. Suppose that the ciphertext DVE CFMV KF NFEUVI, REU KYRK ZJ KYV JVVU FW JTZVETV was produced by encrypting a plaintext message using a shift cipher. What is the original plaintext?

9. Suppose that the ciphertext ERC WYJJMGMIRXPC EHZERGIH XIGLRSPSKC MW MRHMWXMRKYMWLEFPI JVSQ QEKMG was produced by encrypting a plaintext message using a shift cipher. What is the original plaintext?

10. Determine whether there is a key for which the enciphering function for the shift cipher is the same as the deciphering function.

11. What is the decryption function for an affine cipher if the encryption function is $c = (15p + 13)$ **mod** 26?

*12. Find all pairs of integers keys (a, b) for affine ciphers for which the encryption function $c = (ap + b)$ **mod** 26 is the same as the corresponding decryption function.

13. Suppose that the most common letter and the second most common letter in a long ciphertext produced by encrypting a plaintext using an affine cipher $f(p) = (ap + b)$ **mod** 26 are Z and J, respectively. What are the most likely values of a and b?

14. Encrypt the message GRIZZLY BEARS using blocks of five letters and the transposition cipher based on the permutation of $\{1, 2, 3, 4, 5\}$ with $\sigma(1) = 3$, $\sigma(2) = 5$, $\sigma(3) = 1$, $\sigma(4) = 2$, and $\sigma(5) = 4$. For this exercise, use the letter X as many times as necessary to fill out the final block of fewer then five letters.

15. Decrypt the message EABW EFRO ATMR ASIN which is the ciphertext produced by encrypting a plaintext message using the transposition cipher with blocks of four letters and the permutation σ of $\{1, 2, 3, 4\}$ defined by $\sigma(1) = 3$, $\sigma(2) = 1$, $\sigma(3) = 4$, and $\sigma(4) = 2$.

*16. Suppose that you know that a ciphertext was produced by encrypting a plaintext message with a transposition cipher. How might you go about breaking it?

17. Suppose you have intercepted a ciphertext message and when you determine the frequencies of letters in this message, you find the frequencies are similar to the frequency of letters in English text. Which type of cipher do you suspect was used?

The **Vigenère cipher** is a block cipher, with a key that is a string of letters with numerical equivalents $k_1 k_2 \ldots k_m$, where $k_i \in \mathbf{Z}_{26}$ for $i = 1, 2, \ldots, m$. Suppose that the numerical equivalents of the letters of a plaintext block are $p_1 p_2 \ldots p_m$. The corresponding numerical ciphertext block is $(p_1 + k_1)$ **mod** 26 $(p_2 + k_2)$ **mod** 26 $\ldots$ $(p_m + k_m)$ **mod** 26. Finally, we translate back to letters. For example, suppose that the key string is RED, with numerical equivalents 17 4 3. Then, the plaintext ORANGE, with numerical equivalents 14 17 00 13 06 04, is encrypted by first splitting it into two blocks 14 17 00 and 13 06 04. Then, in each block we shift the first letter by 17, the second by 4, and the third by 3. We obtain 5 21 03 and 04 10 07. The cipherext is FVDEKH.

18. Use the Vigenère cipher with key BLUE to encrypt the message SNOWFALL.

19. The ciphertext OIKYWVHBX was produced by encrypting a plaintext message using the Vigenère cipher with key HOT. What is the plaintext message?

20. Express the Vigenère cipher as a cryptosystem.

To break a Vigenère cipher by recovering a plaintext message from the ciphertext message without having the key, the first step is to figure out the length of the key string. The second step is to figure out each character of the key string by determining the corresponding shift. Exercises 21 and 22 deal with these two aspects.

21. Suppose that when a long string of text is encrypted using a Vigenère cipher, the same string is found in the ciphertext starting at several different positions. Explain how this information can be used to help determine the length of the key.

22. Once the length of the key string of a Vigènere cipher is known, explain how to determine each of its characters. Assume that the plaintext is long enough so that the frequency of its letters is reasonably close to the frequency of letters in typical English text.

***23.** Show that we can easily factor n when we know that n is the product of two primes, p and q, and we know the value of $(p-1)(q-1)$.

In Exercises 24–27 first express your answers without computing modular exponentiations. Then use a computational aid to complete these computations.

24. Encrypt the message ATTACK using the RSA system with $n = 43 \cdot 59$ and $e = 13$, translating each letter into integers and grouping together pairs of integers, as done in Example 8.

25. Encrypt the message UPLOAD using the RSA system with $n = 53 \cdot 61$ and $e = 17$, translating each letter into integers and grouping together pairs of integers, as done in Example 8.

26. What is the original message encrypted using the RSA system with $n = 53 \cdot 61$ and $e = 17$ if the encrypted message is 3185 2038 2460 2550? (To decrypt, first find the decryption exponent d, which is the inverse of $e = 17$ modulo $52 \cdot 60$.)

27. What is the original message encrypted using the RSA system with $n = 43 \cdot 59$ and $e = 13$ if the encrypted message is 0667 1947 0671? (To decrypt, first find the decryption exponent d which is the inverse of $e = 13$ modulo $42 \cdot 58$.)

***28.** Suppose that (n, e) is an RSA encryption key, with $n = pq$ where p and q are large primes and $\gcd(e, (p-1)(q-1)) = 1$. Furthermore, suppose that d is an inverse of e modulo $(p-1)(q-1)$. Suppose that $C \equiv M^e \pmod{pq}$. In the text we showed that RSA decryption, that is, the congruence $C^d \equiv M \pmod{pq}$ holds when $\gcd(M, pq) = 1$. Show that this decryption congruence also holds when $\gcd(M, pq) > 1$. [*Hint:* Use congruences modulo p and modulo q and apply the Chinese remainder theorem.]

29. Describe the steps that Alice and Bob follow when they use the Diffie-Hellman key exchange protocol to generate a shared key. Assume that they use the prime $p = 23$ and take $a = 5$, which is a primitive root of 23, and that Alice selects $k_1 = 8$ and Bob selects $k_2 = 5$. (You may want to use some computational aid.)

30. Describe the steps that Alice and Bob follow when they use the Diffie-Hellman key exchange protocol to generate a shared key. Assume that they use the prime $p = 101$ and take $a = 2$, which is a primitive root of 101, and that Alice selects $k_1 = 7$ and Bob selects $k_2 = 9$. (You may want to use some computational aid.)

In Exercises 31–32 suppose that Alice and Bob have these public keys and corresponding private keys: $(n_{\text{Alice}}, e_{\text{Alice}}) = (2867, 7) = (61 \cdot 47, 7)$, $d_{\text{Alice}} = 1183$ and $(n_{\text{Bob}}, e_{\text{Bob}}) = (3127, 21) = (59 \cdot 53, 21)$, $d_{\text{Bob}} = 1149$. First express your answers without carrying out the calculations. Then, using a computational aid, if available, perform the calculation to get the numerical answers.

31. Alice wants to send to all her friends, including Bob, the message "SELL EVERYTHING" so that he knows that she sent it. What should she send to her friends, assuming she signs the message using the RSA cryptosystem.

32. Alice wants to send to Bob the message "BUY NOW" so that he knows that she sent it and so that only Bob can read it. What should she send to Bob, assuming she signs the message and then encrypts it using Bob's public key?

33. We describe a basic key exchange protocol using private key cryptography upon which more sophisticated protocols for key exchange are based. Encryption within the protocol is done using a private key cryptosystem (such as AES) that is considered secure. The protocol involves three parties, Alice and Bob, who wish to exchange a key, and a trusted third party Cathy. Assume that Alice has a secret key k_{Alice} that only she and Cathy know, and Bob has a secret key k_{Bob} which only he and Cathy know. The protocol has three steps:

(i) Alice sends the trusted third party Cathy the message "request a shared key with Bob" encrypted using Alice's key k_{Alice}.

(ii) Cathy sends back to Alice a key $k_{\text{Alice,Bob}}$, which she generates, encrypted using the key k_{Alice}, followed by this same key $k_{\text{Alice,Bob}}$, encrypted using Bob's key, k_{Bob}.

(iii) Alice sends to Bob the key $k_{\text{Alice,Bob}}$ encrypted using k_{Bob}, known only to Bob and to Cathy.

Explain why this protocol allows Alice and Bob to share the secret key $k_{\text{Alice,Bob}}$, known only to them and to Cathy.