Math 145 **Fall 2003**

Working backwards

Examples

Euclid's algorithm.

Given numbers a and b, notice that if we divide a by b and obtain quotient q and remainder r, then sincd $a = qb + r$, the greatest common divisor of a and b is equal to the greatest common divisor of b and r. Euclid's algorithm is based on this fact:

$a = q_1 \cdot b + r_1,$	$r_1 < b,$	$(a, b) = (b, r_1)$	$r_1 = a - q_1 \cdot b$
$b = q_2 \cdot r_1 + r_2,$	$r_2 < r_1,$	$(b, r_1) = (r_1, r_2)$	$r_2 = b - q_2 \cdot r_1$
$r_1 = q_3 \cdot r_2 + r_3,$	$r_3 < r_2,$	$(r_1, r_2) = (r_2, r_3)$	$r_3 = r_1 - q_3 \cdot r_2$
$\ldots \downarrow$	$\ldots$	$\ldots$	$\ldots \uparrow$
$r_{n-2} = q_n \cdot r_{n-1} + r_n,$	$r_n < r_{n-1},$	$(r_{n-2}, r_{n-1}) = (r_{n-1}, r_n)$	$r_n = r_{n-2} - q_n \cdot r_{n-1}$
$r_{n-1} = q_{n+1} \cdot r_n,$	rem. $= 0,$	$(r_{n-1}, r_n) = r_n$	

Thus $(a, b) = r_n$.

Theorem. If $d = (a, b)$, then there exist integer numbers x and y such that $x \cdot a + y \cdot b = d$.

Example. Find the greatest common divisor d of $a = 115$ and $b = 80$, and find x and y such that $x \cdot a + y \cdot b = d$.

Solution.

$115 = 1 \cdot 80 + 35$

$80 = 2 \cdot 35 + 10$

$35 = 3 \cdot 10 + 5$

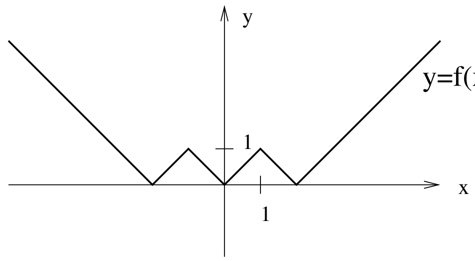
$10 = 2 \cdot 5$

Therefore $(a, b) = 5$. (Note: also, could factor $115 = 5 \cdot 23$, $80 = 2^4 \cdot 5$, so $(115, 80) = 5$.)

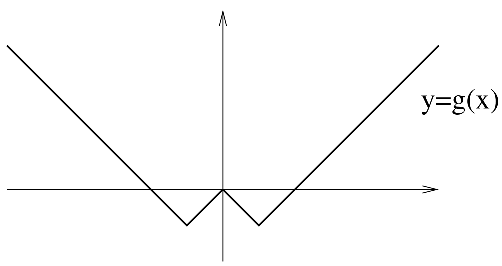
$$\begin{aligned} 5 &= 35 - 3 \cdot 10 \\ &= 35 - 3(80 - 2 \cdot 35) = 35 - 3 \cdot 80 + 6 \cdot 35 = 7 \cdot 35 - 3 \cdot 80 \\ &= 7(115 - 1 \cdot 80) - 3 \cdot 80 = 7 \cdot 115 - 7 \cdot 80 - 3 \cdot 80 = 7 \cdot 115 - 10 \cdot 80 \end{aligned}$$

Thus $x = 7$ and $y = -10$.

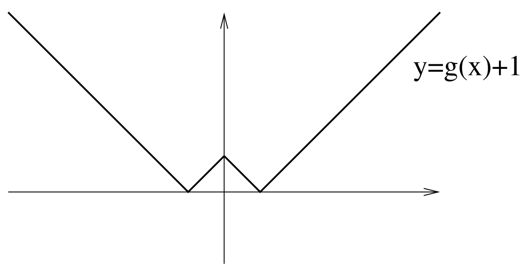
Problem. Find a formula for the function whose graph is shown below.



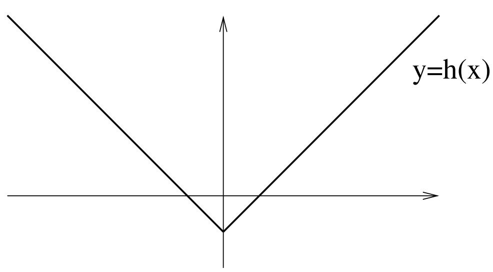
Solution. Let $f(x)$ denote the function that we want to find. Notice that $f(x)$ is the absolute value of the function $g(x)$ whose graph is



(Write it down: $f(x) = |g(x)|$.) Here is the graph of $g(x) + 1$:



Notice that $g(x) + 1$ is the absolute value of $h(x)$ whose graph is

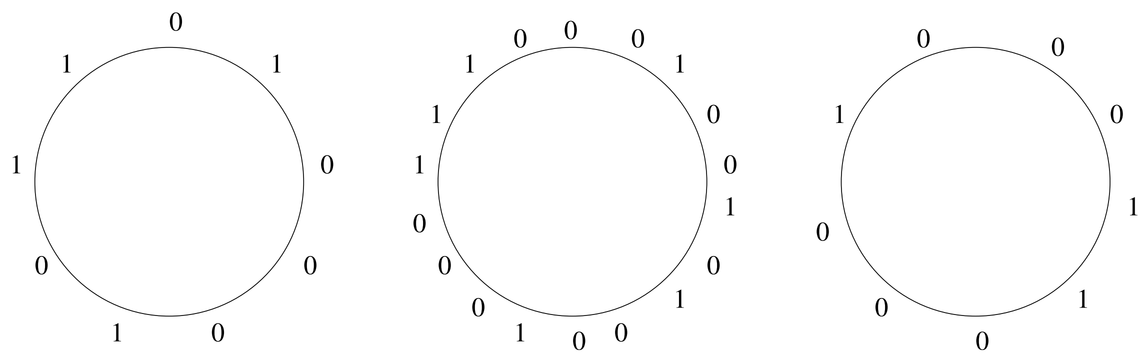


(So, $g(x) + 1 = |h(x)|$.) Finally, the graph of $h(x)$ is obtained from the graph of the absolute value of x by shifting it downward a distance of 1 unit, so $h(x) = |x| - 1$. Now, $g(x) + 1 = |h(x)| = ||x| - 1|$, so $g(x) = ||x| - 1| - 1$, and $f(x) = |g(x)| = |||x| - 1| - 1|$.

Problem.

4 ones and 5 zeros are written along a circle. Then between two equal numbers we write a one and between two distinct numbers zero. Finally the original numbers are wiped out. This step is repeated. Show that we can never reach 9 ones.

A possible initial distribution of ones and zeros and the first step are shown below:



Solution.

Suppose the aim is attainable. Look at the first time we have 9 ones. One step before we must have 9 equal numbers. Since it was the first time we got 9 ones, one step before we must have 9 zeros. Still one step before we have 9 changes $0 - 1 - 0 - 1 - ....$ With an odd number of integers (9), this is not possible.