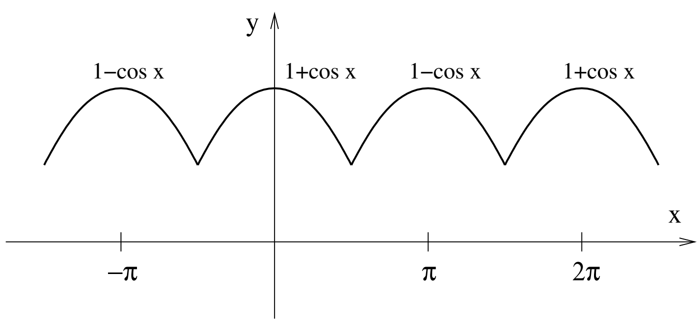
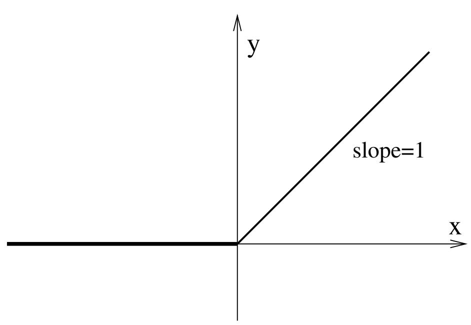
Working backwards

Problems

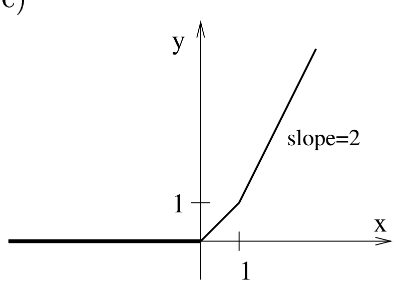
1. Given a and b, find their greatest common divisor $d = (a, b)$, and use Euclid's algorithm to x and y such that $xa + yb = d$.
   (a) $a = 7$, $b = 5$
   (b) $a = 24$, $b = 10$
   (c) $a = 219$, $b = 51$
2. Find a and b such that it will take 5 divisions to reach the greatest common divisor of a and b.
3. Find a formula for the function whose graph is shown.
   (a)



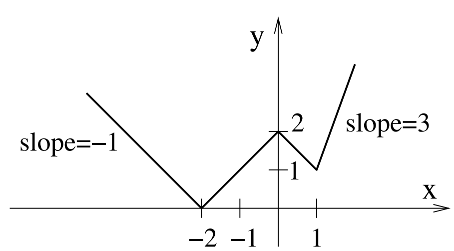
   (b) Hint: try subtracting $\frac{x}{2}$ from the given function.



   (c)



   (d)



4. I have seven coins whose total value is \$0.57. What coins do I have? And, how many of each coin do I have?

5. The integers 1, 2, ..., n are placed in order, so that each value is either bigger than all preceding values or is smaller than all preceding values. In how many ways can this be done?

6. Suppose you want to give your high school students a system of 2 linear equations with 2 variables. You'd like the answers to be integer numbers. You could, of course, try random coefficients, say

$$\begin{cases} 2x + 3y = 4 \\ 5x - 6y = 7 \end{cases},$$

solve your systems, and hope that sooner or later you'll find a system with integer solutions, but is there a better strategy?

7. Two players play the following game.

   - Turns alternate.
   - At each turn, a player removes 1, 2, 3, or 4 counters from a pile that had initially 27 counters.
   - The game ends when all counters have been removed.
   - The player who takes the last counter wins.

   Find a winning strategy for one of the players.

8. Starting from 1, the players take turns multiplying the current number by any whole number from 2 to 9 (inclusive). The player who first names a number greater than 1000 wins. Which player, if either, can guarantee victory in this game?

9. Suppose you are teaching linear algebra, and you need to find matrices with integer entries whose reduced echelon forms also have integer entries. How would you find such matrices?

10. There are two piles of candy. One pile contains 20 pieces, and the other 21. Players take turns eating all the candy in one pile and separating the remaining candy into two (not necessarily equal) piles. (A pile may have 0 candies in it.) The player who cannot eat a candy on his/her turn loses. Which player, if either, can guarantee victory in this game?