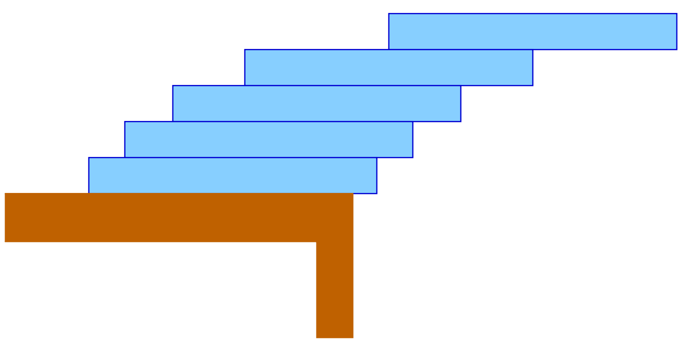
Calculus

Problems

1. (a) Show that the function $f(x) = \ln(x + \sqrt{x^2 + 1})$ is odd.
   (b) Find the inverse of $f(x)$.
2. For which positive numbers a is it true that $a^x \geq 1 + x$ for all x?
3. Find the interval $[a, b]$ for which the value of the integral $\int_a^b (2 + x - x^2)dx$ is a maximum.
4. Suppose you have a large supply of books, all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the one beneath it. Show that it is possible to do this so that the top book extends entirely beyond the table. In fact, show that the top book can extend any distance at all beyond the edge of the table if the stack is high enough. Try the following method of stacking: The top book extends half its length beyond the second book. The second book extends a quarter of its length beyond the third. The third extends one-sixth of its length beyond the fourth, and so on. (You could try it yourself with a deck of cards, or with tapes or CDs.) Consider centers of mass.



5. Find the n-th derivative of the function $f(x) = \dfrac{x^n}{1 - x}$.
6. Find a positive continuous function f such that the area under the graph of f from 0 to t is $A(t) = t^3$ for all $t > 0$.

7. Let $T(x)$ denote the temperature at the point x on Earth at some fixed time. Assuming that T is a continuous function of x, show that at any fixed time there are at least two diametrically opposite points on the equator that have the same temperature.

8. There is a line throught the origin that divides the region bounded by the parabola $y = x - x^2$ and the x-axis into two regions with equal area. What is the slope of that line?

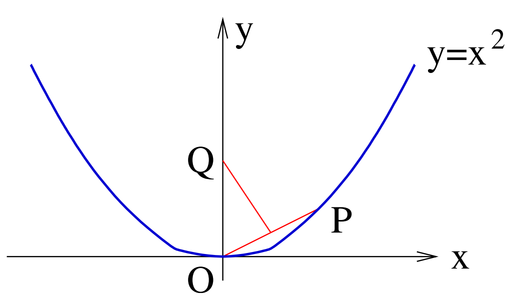
9. Evaluate $\int \frac{1}{x^7 - x} dx$.
   The straightforward approach would be to start with partial fractions, but that would be too brutal. We could reduce the power of the denominator as follows: $\int \frac{1}{x^7 - x} dx = \int \frac{x}{x^8 - x^2} dx$, let $u = x^2$, then $du = 2x dx$, or $\frac{du}{2} = x dx$, and we have $\int \frac{x}{x^8 - x^2} dx = \frac{1}{2} \int \frac{1}{u^4 - u} du$.
   $u^4 - u$ is better than $x^7 - x$, but can you find an even better substitution?

10. Show that any ellipsoid (given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$) has a section that is a circle. Hint: any section of the ellipsoid that passes through the origin is an ellipse.

11. Let a_1, a_2, ..., a_{32} be real numbers. Show that $a_1 \cos x + a_2 \cos(2x) + \ldots + a_30 \cos(30x)$ cannot take on only positive values.

12. Show that, for $x > 0$,
$$\frac{x}{x^2 + 1} < \arctan x < x.$$

13. The figure below shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, what happens to Q? Does it have a limiting position? If so, find it.

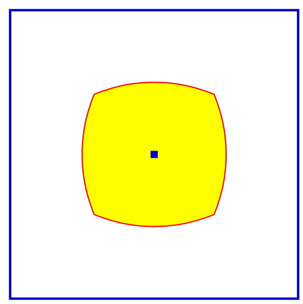


14. If a_0, a_1, a_2, ..., a_k are real numbers and $a_0 + a_1 + a_2 + \ldots + a_k = 0$, show that

$$\lim_{n\to\infty} (a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \ldots a_k\sqrt{n+k}) = 0.$$

Hint: Try the special cases $k = 1$ and $k = 2$ first, and then generalize.

15. The figure below shows a region consisting of all points inside a square that are closer to the center than to the sides of the square. Find the area of the region.

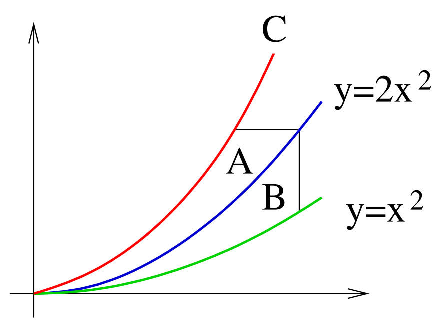


16. Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \ldots$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

17. The figure below shows a curve C with the property that, for every point P on the middle curve $y = 2x^2$, the areas A and B are equal. Find an equation for C.



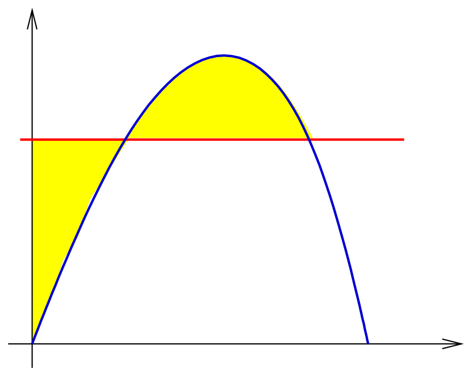
18. Recall that the area of a circle with radius r is $A = \pi r^2$ and the circumference of the circle is $L = 2\pi r$. Notice that $(pir^2)' = 2\pi r$. Similarly, the volume of a ball with radius r is $V = \frac{4}{3}\pi r^3$, the surface area is $S = 4\pi r^2$, and $\left(\frac{4}{3}\pi r^3\right)' = 4\pi r^2$. Is this a coincidence? Actually, it isn't. Explain these facts. What is the ratio of the 4-dimensional volume and the usual 3-dimensional volume of its boundary (the analog of the surface area) for a 4-dimensional ball with radius 4?

19. Find the curve that passes through the point $(3,2)$ and has the property that if the tangent line is drawn at any point P on the curve, then the part of the tangent line that lies in the first quadrant is bisected by P.

20. Evaluate $\int_0^1 (\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3})dx$.

21. Show that e is irrational.

22. The figure below shows a horizontal line $y = c$ intersecting the curve $y = 8x - 27x^3$. Find the number c such that the areas of the shaded regions are equal.



23. Let $f(x) = a_1 \sin x + a_2 \sin(2x) + a_3 \sin(3x) + \ldots + a_n \sin(nx)$, where $a_1, \ldots, a_n$ are real numbers and n is a positive integer. If it is given that $|f(x)| \le |\sin(x)|$ for all x, show that $|a_1 + 2a_2 + \ldots + na_n| \le 1$.

24. Evaluate $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{n}\sqrt{n+n}} \right)$.
Hint: interpret the sum as a Riemann sum of a function. Then the limit as n approaches infinity is the value of an integral.