Math 145 **Fall 2003**

Calculus

Theory (some useful definitions and facts)

Def. $\log_a x = y \quad \Leftrightarrow \quad a^y = x$

Properties of logarithms.

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^r) = r \log_a x$
4. $\log_a(x) = \dfrac{\ln x}{\ln a}$

Def. A function $f(x)$ is called <u>even</u> if $f(-x) = f(x)$ for all x.
$f(x)$ is called <u>odd</u> if $f(-x) = -f(x)$ for all x.

Def. f^{-1} is the <u>inverse</u> of f if

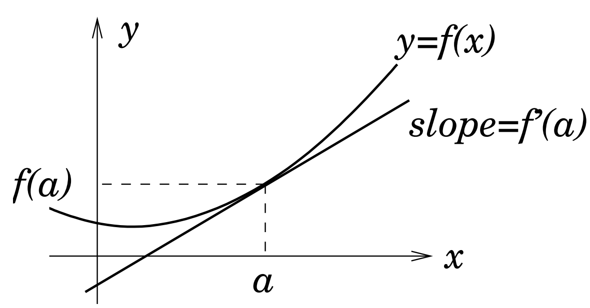
$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad f(x) = y.$$

Intermediate value theorem. Suppose $f(x)$ is continuous on $[a,b]$. Let N be any number between $f(a)$ and $f(b)$. Then there exists $c \in [a,b]$ such that $f(c) = N$.

Def. The derivative of $f(x)$ at a point a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

$f'(a)$ is the slope of the tangent line to $y = f(x)$ at $(a, f(a))$. Also, $f'(a)$ is the rate of change of $f(x)$ with respect to x at $x = a$.



Important derivatives:

$$(x^n)' = nx^{n-1}, \qquad (e^x)' = e^x, \qquad (a^x)' = (\ln a)a^x,$$

$$(c)' = 0, \qquad (\ln x)' = \frac{1}{x}, \qquad (\log_a x)' = \frac{1}{(\ln a)x},$$

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad (\tan x)' = (\sec x)^2,$$

$$(\csc x)' = -\csc x \cot x, \qquad (\sec x)' = \sec x \tan x, \qquad (\cot x)' = -(\csc x)^2,$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \qquad (\arctan x)' = \frac{1}{x^2+1}$$

Chain rule. $(f \circ g)'(x) = (f(g(x)))' = f'(g(x))g'(x)$

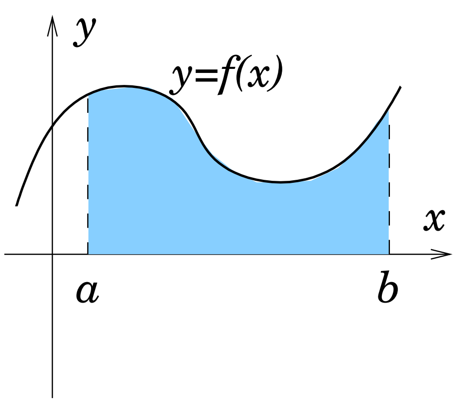
Def. The integral of $f(x)$ from a to b is

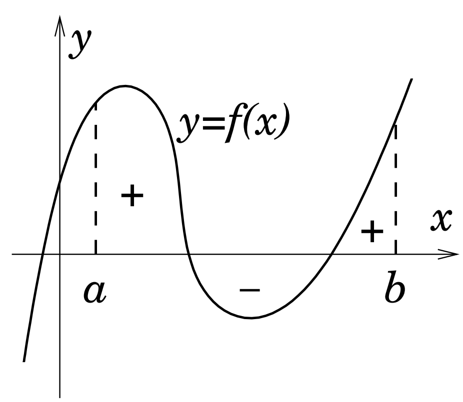
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$ is the right endpoint of the i-th subinterval of $[a, b]$ of length Δx (i.e. the interval $[a, b]$ is divided into n subintervals of equal length).

If $f(x) \geq 0$, then $\int_a^b f(x)dx$ is the area of the region under the curve $y = f(x)$ and above the x-axis from a to b.

If $f(x)$ takes on both positive and negative values, then $\int_a^b f(x)dx$ is the sum of the areas under the curve and above the x-axis minus the sum of the areas under the x-axis and above the curve.





Fundamental Theorem of Calculus.

I. $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$

II. If $F'(x) = f(x)$, then $\int_a^b f(x) = F(b) - F(a)$.

Substitution Rule. $\int f(g(x))g'(x)dx = \int f(u)du$ where $u = g(x)$, $du = g'(x)dx$.

Some important series:

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent.

$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + q^3 + \ldots = \frac{1}{1-q}$ if $|q| < 1$, and divergent if $|q| \geq 1$.

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x$ for all x.

(in particular, if $x = 1$, then $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = e$.)

$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \arctan x$ for all x.

(in particular, if $x = 1$, then $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \arctan 1 = \frac{\pi}{4}$.)