Math 145 **Fall 2003**

Case study

Theory

We have seen that some number theory problems can be solved by considering all possible remainders mod n for a variable. Here are some other typical examples where we need to consider several cases.

If $a^b = 1$, then there are 3 possibilities:

- $a = 1$
- $a \neq 0$, $b = 0$
- $a = -1$ and b is even

Example. Find all the values of x for which $(x^2 - 5x + 5)^{x^2-9x+20} = 1$.

<u>Case I.</u> $a = 1$:
$x^2 - 5x + 5 = 1$
$x = 1$ or $x = 4$

<u>Case II.</u> first solve $b = 0$:
$x^2 - 9x + 20 = 0$
$x = 4$ or $x = 5$
For both roots $a \neq 0$.

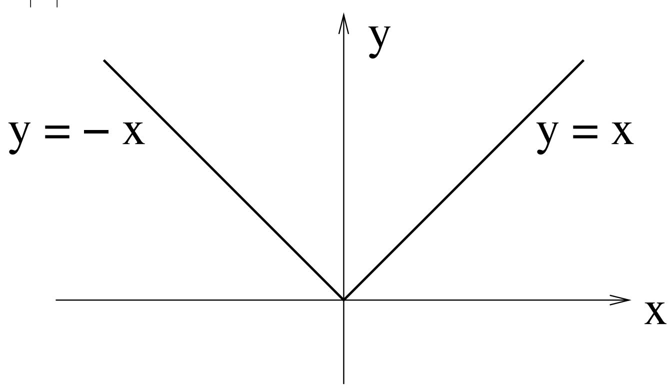
<u>Case III.</u> first solve $a = -1$:
$x^2 - 5x + 5 = -1$
$x = 2$ or $x = 3$
For both roots b is even.

Therefore, the solutions are 1, 2, 3, 4, and 5.

The absolute value of a real number x is denoted $|x|$ and is given by

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Here is the graph of $|x|$:



To solve problems involving the absolute value, consider 2 cases: when the expression inside the absolute value is positive or 0, and when is is negative. If there are several absolute values, consider 2 cases for each absolute value.

Example. Solve $3|x^2 - 9| - 11x + 7 = 0$.

<u>Case I.</u> If $x^2 - 9 \geq 0$, then $|x^2 - 9| = x^2 - 9$, and the equation becomes
$3(x^2 - 9) - 11x + 7 = 0$
$3x^2 - 11x - 20 = 0$
Using the quadratic formula, we find $x = 5$ or $x = -4/3$
$x = 5$ satisfies the condition $x^2 - 9 \geq 0$, but $x = -4/3$ does not, so we throw it away.

<u>Case II.</u> If $x^2 - 9 < 0$, then $|x^2 - 9| = -(x^2 - 9)$, and the equation becomes
$-3(x^2 - 9) - 11x + 7 = 0$
$-3x^2 - 11x + 34 = 0$
the roots are 2 and $-17/3$, but the second root does not satisfy the condition $x^2 - 9 < 0$, so we throw it away.

Thus the only roots are 5 and 2.