Coloring

Examples

Problem. In 1961, the British theoretical physicist M.E.Fisher solved a famous and very tough problem. He showed that an 8×8 chessboard can be covered by 2×1 dominoes in $2^4 \times 901^2 = 12{,}988{,}816$ ways. Now let us cut out two diagonally opposite corners of the board. In how many ways can you cover the 62 squares of the mutilated chessboard with 31 dominoes?

Solution. Zero. There is no way to cover the mutilated chessboard. Each domino covers one black and one white square. If a covering of the board existed, it would cover 31 black and 31 white squares. But the mutilated chessboard has 30 squares of one color and 32 squares of the other color.

Problem. A rectangular floor is covered by 2×2 and 4×1 tiles. One tile got smashed. There is a tile of the other kind available. Show that the floor cannot be covered by rearranging the tiles.

Solution. Color the floor as shown on the picture below. A 4×1 tile always covers either 0 or 2 black squares. A 2×2 tile always covers one black square. Therefore it is impossible to exchange one tile for a tile of the other kind.