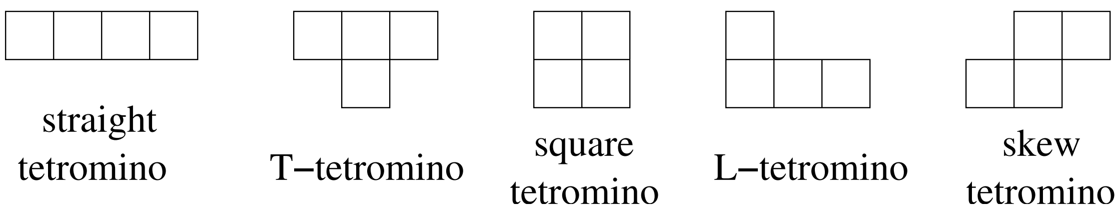
Math 145 **Fall 2003**

Coloring

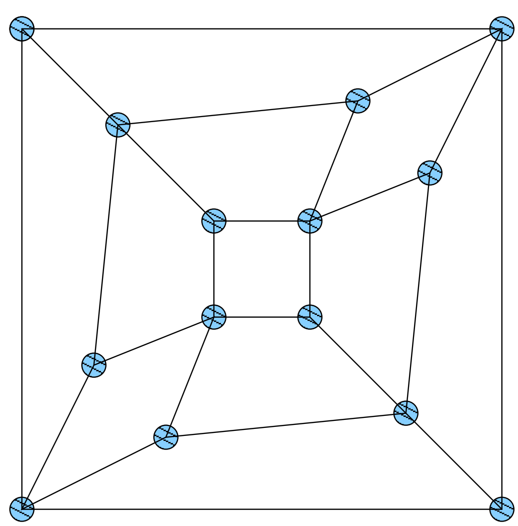
Problems

Chessboard coloring

1. In chess, is it possible for a knight to start at the upper left corner, go throught every square on the chessboard exactly once and reach the lower right corner?
2. Is it possible to form a rectangle with the five tetrominoes shown below?



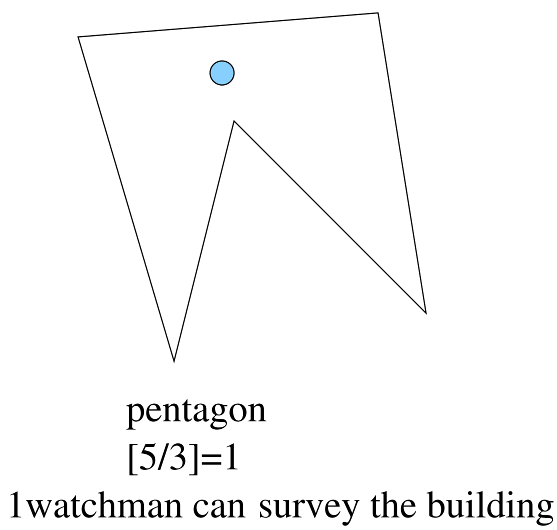
3. Prove that an 8×8 chessboard cannot be covered by 15 T-tetrominoes and one square tetromino.
4. The figure below shows a road map connecting 14 cities. Is there a path passing through each city exactly once?

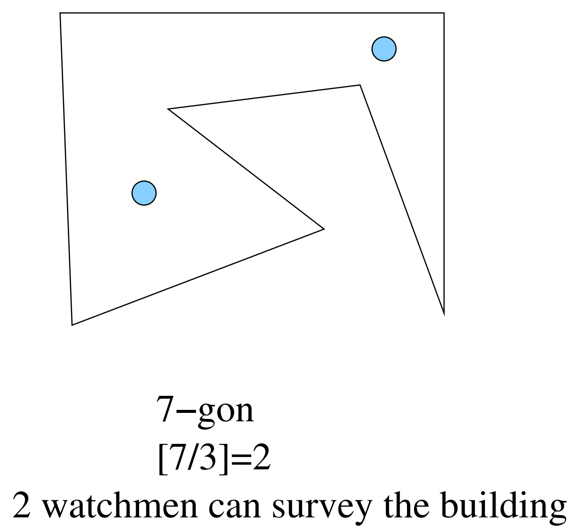


Other colorings

5. Prove that a 10×10 board cannot be covered by 25 straight tetrominoes.
6. Prove that an 8×8 board with one corner square removed (so, 63 squares remain) cannot be covered by 21 straight trominoes (i.e. 3×1 tiles).

7. Prove that a 15×8 board cannot be covered by 2 L-tetrominoes and 28 skew tetrominoes.
8. Prove that a 23×23 square cannot be covered by 2×2 and 3×3 tiles.
9. An $a \times b$ rectangle can be covered by $1 \times n$ rectangles iff $n|a$ or $n|b$.
10. *The Art Gallery Problem.* An art gallery has the shape of an n-gon (not necessarily a convex one). Prove that $[n/3]$ (the integer part of $n/3$) watchmen can survey the building, no matter how complicated its shape. Note 1: the boundary of the n-gon are the only walls, there are no walls inside it. Note 2: We assume that each watchman can turn around and watch in all directions.
e.g.:





3D

11. Prove that there is no way to pack 54 $1 \times 1 \times 4$ bricks into a $6 \times 6 \times 6$ box.

Other problems involving colors, boards, dominoes, etc.

12. The plane is colored with two colors. Prove that there exist three points of the same color, which are vertices of a regular triangle.
13. The plane is colored with n colors where n is any natural number. Prove that there exist four points of the same color, which are vertices of a rectangle. (Hint: recall the "same-color-corner-rectangle" problem.)
14. A 6×6 rectangle is tiled by 2×1 dominoes. Prove that it has at least one *fault-line*, that is, a straight line cutting the rectangle without cutting any domino.
15. Each block of a 25×25 board has either 1 or -1 written on it. Let a_i be the product of all numbers in the ith row and b_j be the product of all numbers in the jth column. Prove that $a_1 + \ldots + a_{25} + b_1 + \ldots + b_{25} \neq 0$.